The Mathematics of Number Puzzles

UEA Summer Research Project

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Last updated: July 22, 2021

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Contents

※ Sudoku

1.1 Shidoku – Smaller 4×4 Grids

To begin, we shall consider the filling-in of a *Shidoku* grid – a variation of the standard Sudoku grid of dimensions 4×4, which is populated in the same fashion as a 9×9 Sudoku grid, but using only the numbers $1 - 4$.

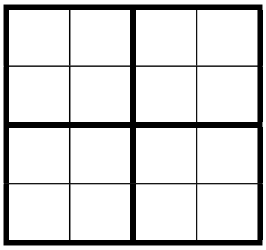

Figure 1.1: *An empty 4×4 Shidoku grid – divided into four blocks of four cells.*

1	2	3	4
3	4	1	2
2	3	4	1
4	1	2	3

Figure 1.2: *An example of a filled-in Shidoku grid.*

1.1.1 Enumeration

In finding the number of valid completions of such a grid, we begin by filling one 2×2 block of the grid, before completing one row of the horizontally adjacent block and one column of the vertically adjacent block – and then by considering the final block.

We have that there are four possible entries for each block (the numbers $1, 2, 3$ and 4), so there are four factorial possible ways to populate the first block we choose to complete. That is, there are $4! = 24$ possible numberings of the blue dots in figure 1.3, below.

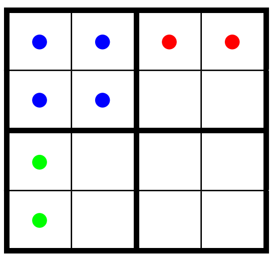

Figure 1.3: *A Shidoku grid labelled for explanation of populating.*

Furthermore, as two of the possible numbers in row and column 1 respectively have been used to fill the blue spaces – there are two ways to number the red dots and two ways to number the green dots for each numbering of the blue dots. Hence, thus far, there are $4! \times 2 \times 2 = 96$ ways to number the cells occupied by coloured dots in figure 1.3.

Now, we must move on to numbering the fourth block (the bottom-right block in figure 1.3). An observation to be made is that whatever number occupies the fourth cell of the first block must also occupy one of each of the green and red dots – and so this number only has one possible position in the fourth block. For example, in figure 1.2, the fourth cell of the first block is occupied by the number 4. It thus appears in the position of one of each of the red and green dots respectively – in this example, it appears in the third cell of the third block and the second cell of the second block. This means that in the fourth block, the number 4 cannot possibly appear in the fourth row or column – as the 4s of these appear in blocks two and three. Hence, the four must appear in cell one of block four.

With one of the four numbers filled in the fourth block, there are three remaining ways to complete the numbering of the rest of that block, based on the restrictions of number placement given by the partial numbering of blocks two and three. All three of these place the number which takes the top-left position of the first block in different positions of the fourth block, and the remaining two numbers can only take one valid place for each placement of this number. This leaves the remaining cells of blocks two and three to be filled in – and there is always only one possible entry left to fill each of these cells as they are all members of a row or column which contains each of the other possible numbers.

Thus, the numbering of the Shidoku grid is completed and we shall now consider how many possibilities there are from this information. We had $4! = 24$ ways to populate the first block, 2 ways to fill in the remaining cells in each of the first row and column respectively for each of the numberings of the first block, 3 ways to number the fourth block per numbering of the first block and first row and column, and a single way to complete the numbering of the grid from here. So, there are

$$\begin{aligned} 4! \times 2 \times 2 \times 3 \times 1 &= 24 \times 2 \times 2 \times 3 \\ &= \mathbf{288\ possible\ filled\ grids}. \end{aligned}$$

1.1.2 Symmetries

For some, leaving it at 288 may suffice, however – for others – it is common to consider results from within group theory to reduce the number of filled grids based on the symmetries of each grid. For example, the two grids shown in figures 1.4 and 1.5 may be considered *"essentially the same"* as they are symmetries of one another by a 180° flip across the vertical centre axis.

4	3	2	1
2	1	4	3
1	4	3	2
3	2	1	4

Figure 1.4: *A filled Shidoku grid, "essentially similar" to that in 1.5 by a flip over the vertical centre axis.*

1	2	3	4
3	4	1	2
2	3	4	1
4	1	2	3

Figure 1.5: *A filled Shidoku grid, "essentially similar" to that in 1.4 by a flip over the vertical centre axis.*

In fact, the symmetries of a Shidoku grid (and also a standard 9×9 Sudoku grid, for that matter), go far beyond those of a square, defined by flips and rotations. Before we go on to look at the symmetries of a Shidoku grid, though, we shall give some definitions.

Definition 1.1. A *band* is a collection of $\sqrt{n}$ horizontally adjacent blocks in an $n \times n$ Sudoku grid, constructed of n blocks. In a Shidoku grid, there are two bands (top and bottom), each containing two blocks (left and right).

Definition 1.2. A *pillar* is a collection of $\sqrt{n}$ vertically adjacent blocks in an $n \times n$ Sudoku grid, constructed of n blocks. In a Shidoku grid, there are two pillars (left and right), each containing two blocks (top and bottom).

Now, with these definitions, we may go on to define the symmetries of a Sudoku grid of any size (including the Shidoku grid).

Definition 1.3 (Rosenhouse and Taalman, 2011)**.** The *symmetries* of a Sudoku grid of any size are given by:

- Relabelling of digits.
- Any rotation by 90°, 180°, 270°, or reflection of 180° across any of the horizontal, vertical, or diagonal axes (as in the dihedral group D_4).
- Permuting the rows in a band.
- Permuting the columns in a pillar.
- Permuting the bands.
- Permuting the pillars.
- Any combinations thereof.

If two symmetries are *essentially the same* (symmetries of each other by the above criteria), then they are considered to be the same numbering. In order to reduce our 288 filled grids down to the number of *essentially different* filled grids, we must consider some definitions and results from group theory.

Definition 1.4 (Cameron, 2008)**.** A *group* is a set G with a binary operation $\circ$ satisfying the following laws:

(G0) ***Closure Law****: For all $g, h \in G$, $g \circ h \in G$.*

(G1) ***Associative Law****: $g \circ (h \circ k) = (g \circ h) \circ k$ for all $g, h, k \in G$.*

(G2) ***Identity Law****: There exists $e \in G$ such that $g \circ e = e \circ g = g$ for all $g \in G$.*

(G3) ***Inverse Law****: For all $g \in G$, $\exists h \in G$ with $g \circ h = h \circ g = e$.*

Definition 1.5. An *action* of a group G on a set X is a function $\mu : G \times X \to X$ (with $\mu(g, x)$ often shortened to $g \cdot x$), which satisfies the following axioms:

(A1) $e \cdot x = x$ *(Identity)*

(A2) $g \cdot (h \cdot x) = (gh) \cdot x$ *(Compatibility).*

for all $g, h \in G$ and $x \in X$.

We say G *acts on* X, and denote this $G \curvearrowright X$.

Definition 1.6 (Jin, 2018)**.** The *orbit* $G.x$ of an element $x \in X$ is the set of elements in X which can be reached from x by the action of some $g \in G$. That is, the set of all possible results of transforming an element x.

$$G.x = \{g.x : g \in G\}.$$

The set of all orbits of X is denoted X/G.

Definition 1.7 (Jin, 2018)**.** For an element $g \in G$, a *fixed point* of X is an element $x \in X$ such that $g.x = x$. That is, x is unchanged by the group action. The elements of X fixed by the action of $g \in G$ are represented by X^g – so there are $|X^g|$ elements $x \in X$ which are fixed by the action of $g \in G$.

Theorem 1.8 (Burnside's Lemma, as stated by Jin, 2018)**.** For a finite group G acting on a set X, the number of orbits of X is equal to the sum of the number of elements fixed by g for every $g \in G$ divided by the cardinality of G. That is:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

In the case of Shidoku grids, our set X is the set of 288 possible grids, and the symmetry group G is the set of symmetries as defined in Definition 1.3. G acts on X. Considering all possible combinations of these symmetries, there are 128 in total. That is, $|G| = 128$. Thus, through application of Burnside's Lemma (1.8), we have that the number of *essentially different* filled Shidoku grids is given by $\frac{1}{128} \sum_{g \in G} |X^g|$.

Through brute force computations, Elizabeth Arnold and Stephen Lucas (cited in Rosenhouse and Taalman, 2011) discovered that of the 128 symmetries, 56 fix no grids in X, 48 fix two grids, 9 fix four, 4 fix six, 6 fix eight, 4 fix ten and 1 fixes twelve. Substituting these values into Burnside's Lemma (1.8), we have:

$$\frac{1}{128} \times ((56 \times 0) + (48 \times 2) + (9 \times 4) + (4 \times 6) + (6 \times 8) + (4 \times 10) + (1 \times 12)) = 2.$$

Hence, we have that there are just two essentially different Shidoku grid completions up to symmetry:

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

1	2	3	4
3	4	1	2
2	3	4	1
4	1	2	3

That is, each of the 288 possible Shidoku grids are in the orbit of exactly one the grids above – and both are in different orbits, so cannot be reached from each other by the action of any $g \in G$ – as can be verified in GAP, using the code shown in Appendix A.

	B1			B2			B3	
	B4			B5			B6	
	B7			B8			B9	

Figure 1.6: *An empty 9×9 Sudoku grid with block labels – divided into nine blocks of nine cells.*

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	1	5	6	4	8	9	7
5	6	4	8	9	7	2	3	1
8	9	7	2	3	1	5	6	4
3	1	2	6	4	5	9	7	8
6	4	5	9	7	8	3	1	2
9	7	8	3	1	2	6	4	5

Figure 1.7: *An example of a fully numbered Sudoku grid.*

1.2 The Standard Size 9×9 Grid

The more popular *standard-sized* Sudoku grid is 9×9 in dimension, made up of 9 blocks of 9 cells, arranged into 3×3 grids (Figures 1.6 and 1.7). This is filled-in with numbers $1 - 9$ appearing exactly once in each block, row and column respectively.

Notation 1.9. We shall refer to the blocks of a Sudoku grid as Bx, where x is the block number as defined in Figure 1.6. For example, the bottom-right block shall be denoted $B9$.

1.2.1 Enumeration

Much like with the 4×4 Shidoku grid, the typical method of counting possible numberings of a Sudoku grid begins by considering a first block and stemming out from this.

Starting with block $B1$, there are $9! = 362880$ possible ways to populate this block. As described by Felgenhauer and Jarvis (2005), it then follows to consider the filling-in of the first rows of $B2$ and $B3$. Felgenhauer and Jarvis (2005) shows that there are 20 possible configurations of numbers (without regard to order) for the top rows of $B2$ and $B3$: two *pure* and eighteen *mixed*.

Definition 1.10. A band or pillar contains *pure* rows or columns if the rows or columns of the blocks in that band or pillar respectively contain permutations of the same elements as one another.

For example, all bands and pillars of the grid in Figure 1.7 are constructed of *pure* rows and columns – as, for example, the numbers $[1, 2, 3]$ stick together in the same row as each other in each block of the first band. Similarly, the same holds for all $[x, y, z]$ which are together in any row or column in any block, and the blocks of the corresponding band or pillar.

Definition 1.11. A band or pillar contains *mixed* rows or columns if they are not pure by Definition 1.10.

For the pure top rows, there are $(3!)^6$ possible completions of the blocks $B2$ and $B3$ – because there are 3! ways of arranging the three numbers in each of the three rows of each of the two blocks, all of which are independent from one another. For the mixed top rows, there are significantly more possible numberings, though. The top row can be fully determined by the choice of mixed numbering (there are 18 of these), but in each of the blocks there will be one row with just one determined number two cells with multiple completions, and another row with two determined numbers and one cell with multiple possible completions. The completion of the unknown cells in each block are codependent on each other – so there are three possible completions. Thus, for the mixed top rows, there are $3 \times (3!)^6$ possible configurations. So, we have

$$2 \times (3!)^6 + 18 \times 3 \times (3!)^6 = 2612736$$

possible ways to populate the top band ($B1$, $B2$, and $B3$).

Blocks $B4$ and $B7$ can be populated in exactly the same way, and so we may say that for each of the 2612736 numberings of the first (top) band, there are 2612736 completions of the first (left-most) pillar.

The *inner loop*, containing blocks $B5$, $B6$, $B8$, and $B9$, can be filled-in by computer – as was done by Felgenhauer and Jarvis (2005). This computation yields a result that there are $6\,670\,903\,752\,021\,072\,936\,960 \approx 6.671 \times 10^{21}$ valid numberings of the 9×9 Sudoku grid. This number, denoted N_0, includes all numberings though – even ones which are *essentially the same* by symmetry.

1.2.2 Symmetries

Similarly to the method of reducing Shidoku grids to those essentially different (according to the action of the same symmetry group defined in Definition 1.3), Burnside's lemma may also be utilised for N_0. Of course, with much larger numbers, larger sets and exponentially more symmetries, these computations are not simple and so are found through computer programs. Russell and Jarvis (2006) carried out these computations, reducing N_0 to 5 472 730 538 *essentially different* filled 9×9 grids.

1.3 Clues

Clues in Sudoku puzzles are pre-numbered cells which serve to assist the solver in completing the puzzle. A 9×9 puzzle with 0 clues has 6 670 903 752 021 072 936 960 solutions (all possible filled grids) – as, with no pre-numbered cells, any valid completed grid would be a valid solution.

Definition 1.12. A *solution* S_i is a valid filled Sudoku grid.

Definition 1.13. An *instance* I_i is a Sudoku grid of clues which are laid out in a valid form, from which at least one solution can be acquired. $|I_i|$ is the number of clues in an instance I_i.

Definition 1.14. A *proper* instance is one which generates a unique solution. Otherwise, it is *improper*.

Definition 1.15. A *minimal* instance is a proper instance which becomes improper on the removal of any clue.

In this section, we will review claims and proofs that there exist no *proper* instances with fewer than 17 clues. This does not however mean that all instances with 17 or more clues are proper, however. For example, Figure 1.8 shows an improper instance with 77 clues.

1	2	3	4	5	6	7		
4	9	8	3	7	2	5	6	1
5	7	6	1	8	9	2	4	3
3	8	2	6	9	5	4	1	7
9	4	5	7	2	1	8	3	6
6	1	7	8	4	3	9	2	5
8	5	1	9	3	4	6	7	2
7	6	9	2	1	8	3	5	4
2	3	4	5	6	7	1		

Figure 1.8: *An improper 9×9 Sudoku instance with 77 clues. The remaining cells in the top row can be validly filled-in with 8 and 9 in any order, and the remaining cells in the bottom row can be validly filled-in with 8 and 9 in the reverse of the order chosen for the top row.*

Lemma 1.16. For an $n \times n$ Sudoku grid, the largest improper instance has $|I_i| = n^2 - 4$.

Proof. An instance may be constructed such that all cells are filled-in but four, of which two must lie horizontally adjacent and contained in one block, and the remaining two must lie in the same columns as the first two, but in a different block. The numbering of the two rows (and blocks) which are occupied by the empty cells should be permutations of one another – meaning that both sets of empty cells can be filled with the same two numbers – i and j, for example. If the first two empty cells are numbered $[i, j]$, then the remaining must be numbered $[j, i]$. Equally valid is the solution where the first two empty cells are numbered $[j, i]$ and the remaining are numbered $[i, j]$. Hence, there are two solutions for any instance I_i of an $n \times n$ Sudoku grid with $|I_i| = n^2 - 4$ – so I_i is improper. $\square$

Corollary 1.17. All instances of an $n \times n$ Sudoku grid with $|I_i| \geq n^2 - 3$ are proper.

Proof. In all possible configurations of three empty cells, at least two of them are members of a row or column which contains eight filled cells – so there exists only one possible numbering for each of these cells. The remaining cell also only has one possible numbering – thus completing the board and generating a solution. $\square$

Proposition 1.18. The smallest proper instance of a 4×4 Shidoku puzzle has 4 clues.

Proof. We know there exists a proper instance with 4 clues (see Figure 1.9), so we shall show that any 4-clue proper instance is minimal. That is, there exists no 3-clue proper instance for a Shidoku grid. Suppose we have a partial Shidoku with three pre-filled cells. the first filled cell is in the position $(1, 1)$ – the top-left cell. Let this cell be occupied by 1. Let the second filled cell be occupied by 2. There are three cases for where the second filled cell may be.

Case 1: The second filled cell is in the first row or column – $(1, j)$ or $(i, 1)$ respectively,

1			
		2	
	3		
			4

Figure 1.9: *A proper Shidoku instance with 4 clues.*

In this case, if the third filled cell is in this same row/column as the first two, there exist two adjacent empty blocks whose numberings are ambiguous – and their rows/columns may be swapped. So three clues in the same row/column make an improper instance.

In the case that the third clue is not in the same COME BACK TO THIS. https://theory.tifr.res.in/~sgupta/sudoku/theorems.pdf

Case 2:

Case 3:

$\square$

https://web.archive.org/web/20060102111254/http://www.csse.uwa.edu.au/~gordon/sudokumin.php

https://theory.tifr.res.in/~sgupta/sudoku/shidoku.html

1.3.1 Valid Instances Vs. Valid Solutions

Whilst numbers can be added to an empty Sudoku grid in a seemingly valid way, not all can generate a valid solution – so not all are *instances*. Similarly, many can generate multiple solutions. In this subsection, we shall explore the proportion of valid solutions per valid instance.

Lemma 1.19. An upper bound for the number of valid instances for an $n \times n$ Sudoku puzzles is $(n+1)^{n^2}$.

Proof. There are n^2 cells in a Sudoku grid, for each of which there are $n+1$ choices of their status in an instance. They can contain one of the numbers $1, 2, \cdots, n$, or they can remain blank. Therefore, with $n+1$ choices for each of the n^2 cells – there are $(n+1)^{n^2}$ possible configurations from this. This includes cases which do not generate valid instances, though – for example, where cells housed in the same block, row or column are filled-in identically – which is invalid in the context of Sudoku. Hence, the number of instances for an $n \times n$ Sudoku grid is strictly less than $(n+1)^{n^2}$. $\square$

In the case of Shidoku (4×4 Sudoku grids), this means that there are strictly less than $5^{16} = 152\,587\,890\,625$ valid instances. This can, in fact, be broken down further by considering binomial coefficients.

Definition 1.20. The *binomial coefficient* $\binom{n}{k}$ is the number of ways of choosing k objects from a set of n objects.

We can utilise this to say that for an $n \times n$ Sudoku grid, there are $\binom{n}{k}$ ways to select k cells to fill-in in an instance I_i with $|I_i| = k$, and – by the same argument used in Lemma 1.19 – there are at most 4^k instances per set of k cells.

To be completed...

http://magictour.free.fr/sudoku.htm

1.4 Other Interesting Facts about Sudoku

Theorem 1.21 (Phistomefel's Theorem)**.** The digits in the cells labelled $x_{i,j}$ (in Figure 1.10) are the same as those in the cells labelled $y_{i,j}$.

$x_{1,1}$	$x_{1,2}$	•	•	•	•	•	$x_{1,8}$	$x_{1,9}$
$x_{2,1}$	$x_{2,2}$	•	•	•	•	•	$x_{2,8}$	$x_{2,9}$
•	•	$y_{3,3}$	$y_{3,4}$	$y_{3,5}$	$y_{3,6}$	$y_{3,7}$	•	•
•	•	$y_{4,3}$				$y_{4,7}$	•	•
•	•	$y_{5,3}$				$y_{5,7}$	•	•
•	•	$y_{6,3}$				$y_{6,7}$	•	•
•	•	$y_{7,3}$	$y_{7,4}$	$y_{7,5}$	$y_{7,6}$	$y_{7,7}$	•	•
$x_{8,1}$	$x_{8,2}$	•	•	•	•	•	$x_{8,8}$	$x_{8,9}$
$x_{9,1}$	$x_{9,2}$	•	•	•	•	•	$x_{9,8}$	$x_{9,9}$

Figure 1.10: *A 9×9 Sudoku grid, labelled for explanation of Phistomefel's Theorem (Theorem 1.21).*

In order to prove Phistomefel's Theorem, we shall first make the following observations:

Lemma 1.22. In a standard-size 9×9 Sudoku grid, the digits in each row, column and block respectively sum to 45.

Proof. Each row, column and block contains the numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}(= [9])$ in some order.

$$\sum_{n \in [9]} n = 45.$$

□

Notation 1.23. Let x be the sum of digits labelled $x_{i,j}$ in 1.10, and let y be the sum of the digits labelled $y_{i,j}$ in 1.10. That is,

$$x = x_{1,1}+x_{1,2}+x_{1,8}+x_{1,9}+x_{2,1}+x_{2,2}+x_{2,8}+x_{2,9}+x_{8,1}+x_{8,2}+x_{8,8}+x_{8,9}+x_{9,1}+x_{9,2}+x_{9,8}+x_{9,9},$$

$$y = y_{3,3}+y_{3,4}+y_{3,5}+y_{3,6}+y_{3,7}+y_{4,7}+y_{5,7}+y_{6,7}+y_{7,7}+y_{7,6}+y_{7,5}+y_{7,4}+y_{7,3}+y_{6,3}+y_{5,3}+y_{4,3}.$$

Lemma 1.24. For x, y as defined above (Notation 1.23), $x = y$.

Proof. If we wished to find the sum of the digits in the outer two rings (marked red in Figure 1.10) of a 9×9 Sudoku grid, there are two ways to do this:

Method 1: We may take the sum of the outer blocks ($B1, B2, B3, B4, B6, B7, B8, B9$), which – by Lemma 1.22 – is equal to

$$8 \times 45 = 360,$$

and subtract the sum of the cells labelled $y_{i,j}$, which is equal to y. Hence, the sum of the values in cells of the two outer rings is

$$360 - y.$$

Method 2: Alternatively, we may instead take the sum of the digits in rows and columns 1, 2, 8 and 9 (each of which is equal to 45). The summation of the rows and columns mentioned is equal to

$$8 \times 45 = 360.$$

However, we must recognise that the cells labelled $x_{i,j}$ in Figure 1.10 are counted twice in this summation – so their values must be subtracted.

That is, the sum of the values in the cells of the two outer rings is equal to

$$360 - x.$$

Hence, we have shown that $360 - y = 360 - x$ – and thus $x = y$.

□

Proof of Phistomefel's Theorem (Thm 1.21). **Finish this.** □

※ Futoshiki

2.1 Latin Squares (in relation to Futoshiki)

Definition 2.1. A *Latin Square* is an $n \times n$ grid, populated with the numbers $1 - n$, each of which appear precisely n times – exactly once in each row and column of the grid.

2.1.1 Enumeration

We shall first focus our attention on the simple question of how many $n \times n$ Latin squares exist. First, we must define some concepts though.

Notation 2.2 (Shao and Wei, 1992)**.** B_n is the set of all $n \times n$ matrices with entries in the set $\{0, 1\}$.

Notation 2.3 (Shao and Wei, 1992)**.** $\sigma_0(A)$ is the number of zero entries in a matrix A.

Definition 2.4. The *permanent* $\mathrm{per}(A)$ of an $n \times n$ matrix $A = (a_{i,j})$ is given by

$$\mathrm{per}(A) := \sum_{s \in S_n} \prod_{i=1}^{n} A_{i,s(i)}.$$

where S is the set of all permutations of $[1, 2, \cdots, n]$.

Example 2.5. For a matrix $a = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the permanent $\mathrm{per}(A) = ad + bc$.

Theorem 2.6 (Shao and Wei, 1992)**.** The number L_n of $n \times n$ Latin squares is given by

$$L_n = n! \sum_{A \in B_n} (-1)^{\sigma_0(A)} \binom{\mathrm{per}(A)}{n}.$$

Before proving this, we shall first give some definitions:

Definition 2.7. An $n \times n$ *permutation matrix* is a matrix whose rows can be permuted to reach the $n \times n$ identity matrix. That is, a *permutation matrix* is a matrix whose entries are 0 and 1, with precisely one 1 in each row and column.

Example 2.8. The following matrices are all examples of 3×3 permutation matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Proof of Theorem 2.6 (Shao and Wei, 1992). Let S_n be the set of $n \times n$ permutation matrices and let S be the set of all $n-$permutations of the elements in the set S_n. Define the following n^2 subsets of S:

$$S_{ij} = \{(P_1, \cdots, P_n) \in S : (P_1)_{ij} = \cdots = (P_n)_{ij} = 0\} \qquad (1 \leq i, j \leq n)$$

where $(A)_{ij}$ denoted the $(i, j)-$entry of A.

Let J_n be the $n \times n$ matrix with all elements 1.

Every letter $i \in \{1, 2, \cdots, n\}$ in a Latin square determines a permutation matric P_i, whose non-zero positions correspond to the positions occupied by i in the given Latin square. Therefore, a Latin square of order n is equivalent to an ordered set of n distinct permutation matrices $(P_1, P_2, \cdots, P_n) \in S$ satisfying $P_1 + \cdots P_n = J_n$. On the other hand,

$$\begin{aligned} P_1 + \cdots P_n = J_n &\iff (P_1)_{ij}, \cdots (P_n)_{ij} \text{ are not all zero, for all } 1 \leq i, j \leq n \\ &\iff (P_1, \cdots, P_n) \in \cap_{1 \leq i,j \leq n} \overline{S_{ij}}. \end{aligned}$$

Hence, we have

$$L_n = \left| \bigcap_{1 \leq i,j \leq n} \overline{S_{ij}} \right|$$

For any $X \subseteq I_n \times I_n$, let

$$S_X = \cap_{(i,j) \in X} S_{ij}.$$

Then, by the inclusion-exclusion principle, we have

$$L_n = \sum_{X \subseteq I_n \times I_n} (-1)^{|X|} |S_X|. \tag{1}$$

We now need a formula for $|S_X|$. Let $A(X)$ be the $(0, 1)$ matrix of order n with 0's in the positions of X and 1's in the positions not in X. Then,

$$\begin{aligned} (P_1, \cdots, P_n) \in S_X &\iff (P_1)_{ij} = \cdots (P_n)_{ij} = 0 \text{ for all } (i, j) \in X \\ &\iff (P_1, \cdots, P_n) \in S \text{ and } P_1 \leq A(X), \cdots, P_n \leq A(X). \end{aligned}$$

Let $P_X = \{P \in S_n : P \leq A(X)\}$ be the set of $n \times n$ permutation matrices contained in $A(X)$. Then,

$$|P_X| = \text{per}(A(X))$$

Now, S_X is just the set of $n-$permutations of the elements in P_X. So,

$$|S_X| = n! \binom{|P_X|}{n} = n! \binom{\text{per}(A(X))}{n}. \tag{2}$$

Combining (1) and (2), and substituting $A(X) = A$, we have

$$L_n = n! \sum_{A \in B_n} (-1)^{\sigma_0(A)} \binom{\text{per}(A)}{n}.$$

□

A standard-sized Futoshiki puzzle is 5×5 in dimension, so we may use Theorem 2.6 to compute the number of 5×5 Latin squares. First, we must compute $|B_5|$. This is simple, as there are 25 entries in a 5×5 matrix, and there is a binary choice of number for each of these entries. Hence, there are $2^{25} = 33\,554\,432$ possible 5×5 matrices with entries in $\{0, 1\}$, so $|B_5| = 33\,554\,432$.

From here, the process complicates heavily – and so calculations should be done by a computer. A result of 161 280 is held by OEIS Foundation Inc. (2004) for the number of 5×5 Latin squares. However, just like with Sudoku grids, some may wish to consider symmetries of certain Latin squares too – in order to reduce this number.

2.1.2 Symmetries

For Latin Squares, the notions of *structurally distinct* and *isotopic* numberings are relevant when seeking symmetries and reducing the amount of numerations as such. As we are considering Latin squares in relation to Futoshiki puzzles, due to the importance of the cells adjacent to each cell in a Futoshiki puzzle, the operations of permuting rows and columns and relabelling entries do not preserve the uniqueness of a puzzle (unlike with traditional Latin squares, considered separately from Futoshiki) – as Futoshiki heavily depends on the inequality relations between entries and their adjacent cells. Thus, we should consider G acting on[1] the set of Latin squares X by the natural action of the dihedral group D_4 on the square induced by X – as the symmetries of D_4 preserve the uniqueness of a Futoshiki puzzle. As noted by Barink (2015), there are at most eight symmetries to each Latin square when acted on by D_4 – each generated by a rotation of $90°$, and/or by a flip across any axis.

Definition 2.9. Two Latin squares are *structurally distinct* if neither is a renumbering, reflection, rotation, or combination thereof, of the other. Otherwise, they are *structurally identical.*

Definition 2.10. Two Latin squares are *isotopic* if each can be turned into the other by permuting the rows, columns, and symbols.

Definition 2.11. $R(L_n)$ is the number of *reduced* Latin squares of dimension n. When reduced, the first row and column respectively of a square of dimension n is $[1, 2, \cdots, n]$.

To reduce our 161 280 possible 5×5 Latin squares, we must divide by $5! \times 4!$. That is, we must divide by the number of permutations of the first column, multiplied by the number of permutations of the first row with the first column fixed. In general, for an $n \times n$ Latin square, we have

$$R(L_n) = \frac{L_n}{n!(n-1)!}.$$

As held by OEIS Foundation Inc. (1995), for a 5×5 Latin square, there are 56 reduced squares.

Notation 2.12. Let f_{d_1} denote a flip (or reflection) across the diagonal running from the top-left to the bottom-right of a Latin square. Also, let f_{d_2} denote a flip (or reflection) across the central-diagonal which is orthogonal to d_1.

For our set X of all $n \times n$ Latin squares, we shall consider the action of the symmetry group $G = D_4$ ($D_4 \curvearrowright X$) which contains the identity element e, rotations $r_{90}, {r_{90}}^2, {r_{90}}^3$, and flips f_h and f_v across the horizontal and vertical axes respectively, and flips f_{d_1} and f_{d_2} across the two diagonal axes respectively. That is,

$$G =< r_{90}, f_h >= \{e, r_{90}, {r_{90}}^2, {r_{90}}^3, f_h, f_v, f_{d_1}, f_{d_2}\}.$$

Proposition 2.13. For any $n \times n$ Latin square $x \in X$ of odd order (n is odd), the group action of rotating by $90°$, r_{90}, has order 4.

[1]See Definition 1.5

Proof. For contradiction, suppose that r_{90} does not have order 4. We know ${r_{90}}^4 = e$ – the identity. Thus, $o(r_{90}) > 4$, so the order of r_{90} must be 1, 2, or 3.

Case 1: $o(r_{90}) = 1 \iff r_{90} = e$.

$$x = \begin{matrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{matrix} \xrightarrow{r_{90}} \begin{matrix} x_{n,1} & \cdots & x_{1,1} \\ \vdots & \ddots & \vdots \\ x_{n,n} & \cdots & x_{1,n} \end{matrix}$$

As $r_{90} = e$, we must have $x_{1,1} = x_{1,n} = x_{n,n} = x_{n,1}$. But if this is the case, then x is not a Latin square – as it would have two of the same number in the first and last rows and columns. Thus, a contradiction.

Case 2: $o(r_{90}) = 2 \iff {r_{90}}^2 = e$.

$$x = \begin{matrix} x_{1,1} & \cdots & x_{1,\lceil\frac{n}{2}\rceil} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{\lceil\frac{n}{2}\rceil,1} & \cdots & x_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil} & \cdots & x_{\lceil\frac{n}{2}\rceil,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,\lceil\frac{n}{2}\rceil} & \cdots & x_{n,n} \end{matrix} \xrightarrow{{r_{90}}^2} \begin{matrix} x_{n,n} & \cdots & x_{n,\lceil\frac{n}{2}\rceil} & \cdots & x_{n,1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{\lceil\frac{n}{2}\rceil,n} & \cdots & x_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil} & \cdots & x_{\lceil\frac{n}{2}\rceil,1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{1,n} & \cdots & x_{1,\lceil\frac{n}{2}\rceil} & \cdots & x_{1,1} \end{matrix}$$

As ${r_{90}}^2 = e$, this implies that the ij^{th} entries in x are equal to those in the rotation, on the right. If this is the case, then $x_{1,\lceil\frac{n}{2}\rceil} = x_{n,\lceil\frac{n}{2}\rceil}$ and $x_{\lceil\frac{n}{2}\rceil,1} = x_{\lceil\frac{n}{2}\rceil,n}$ – but this is impossible, as it would mean that there exists two equal entries in both row $\lceil\frac{n}{2}\rceil$ and column $\lceil\frac{n}{2}\rceil$. Thus, x cannot be a Latin square, and we have a contradiction.

Case 3: $o(r_{90}) = 3 \iff {r_{90}}^3 = e$.

$$x = \begin{matrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{matrix} \xrightarrow{{r_{90}}^3} \begin{matrix} x_{1,n} & \cdots & x_{n,n} \\ \vdots & \ddots & \vdots \\ x_{1,1} & \cdots & x_{n,1} \end{matrix}$$

As ${r_{90}}^3 = e$, we must have $x_{1,1} = x_{1,n} = x_{n,n} = x_{n,1}$. But if this is the case, then x is not a Latin square – as it would have two of the same number in the first and last rows and columns. Thus, a contradiction.

Hence, with a contradiction in all three cases, it must be true that $r_{90} \in G$ acting on X (rotating by 90°), has order 4. □

Corollary 2.14. As a result of Proposition 2.13, we have that the actions of rotating an $n \times n$ Latin square (with n odd) by 180° (${r_{90}}^2$) and 270° (${r_{90}}^3$) have orders 2 and 4 respectively.

Note: *From this point onward, we will only consider Latin squares of odd order, and more specifically – as it is the most common dimension of Futoshiki puzzle – the* 5×5 *Latin square.*

Proposition 2.13 tells us that every odd $n \times n$ Latin square has 4 unique symmetries by the operation of rotation by 90° – which implies that none of the rotation elements of the dihedral group D_4 ($r_{90}, {r_{90}}^2, {r_{90}}^3$) fix any grids. Next, we must consider reflections (or flips) across centre axes.

Proposition 2.15. A Latin square cannot be symmetrical across a horizontal or vertical centre-axis.

Proof. For contradiction, suppose a Latin square can be symmetrical across its horizontal (case 1) or vertical (case 2) axis.

Case 1: The Latin square x is symmetrically identical across its horizontal centre-axis.

$$x \quad = \quad \begin{matrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{matrix}$$

As the Latin square is symmetrical about its horizontal axis, we have $x_{1,1} = x_{n,1}$ and $x_{1,n} = x_{n,n}$ – but this is absurd, as two entries in the same column cannot be equal in a Latin square. Thus, x is NOT a Latin square – giving a contradiction.

Case 2: The Latin square y is symmetrical across its vertical centre-axis.

$$y \quad = \quad \begin{matrix} y_{1,1} & \cdots & y_{1,n} \\ \vdots & \ddots & \vdots \\ y_{n,1} & \cdots & y_{n,n} \end{matrix}$$

As the Latin square is symmetrical about its horizontal axis, we have $y_{1,1} = y_{1,n}$ and $y_{n,1} = y_{n,n}$ – but this is absurd, as two entries in the same row cannot be equal in a Latin square. Thus, y is NOT a Latin square – giving a contradiction. □

As a result of Proposition 2.15, we can say that the order of reflections across either of the horizontal (f_h) or vertical (f_v) axes is 2. That is, a Latin square cannot be symmetrical across its vertical or horizontal centre-axis. Latin squares can, however, be symmetrical across a diagonal axis – as illustrated in Figure 2.1.

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \\ 3 & 4 & 2 & 5 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 5 & 3 & 1 & 2 & 4 \end{matrix} \quad \xrightarrow{f_{d_1}} \quad \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \\ 3 & 4 & 2 & 5 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 5 & 3 & 1 & 2 & 4 \end{matrix}$$

Figure 2.1: *An example of a 5×5 Latin square which is symmetrical about the diagonal axis d_1 – meaning that, for this particular Latin square, the order $o(f_{d_1})$ is 1 – because f_{d_1} acts as the identity.*

In order to proceed with Burnside's Lemma (Theorem 1.8), we must consider how many of the 161 280 Latin squares of dimension 5 are fixed by the group actions f_{d_1} and f_{d_2} respectively (it is trivial that all 161 280 are fixed by e, and we have shown that none are fixed by $f_h, f_v, r_{90}, {r_{90}}^3$, and – when n is odd – ${r_{90}}^2$ – as all of these symmetries have order above 1). In order to help with this, we can make two observations:

Proposition 2.16. Let X be the set of all 5×5 Latin squares. Then $X^{f_{d_1}} = X \cap \mathrm{Sym}_5$ – where Sym_5 is the set of all 5×5 symmetric matrices.

Proof. For an Latin square $x \in X$ (where X is the set of all 5×5 Latin squares) to be fixed by f_{d_1}, the ij^{th} entry must be equal to the ji^{th} entry, for all $1 \leq i, j \leq n$ – which is the definition of a symmetric matrix. □

Lemma 2.17. $|X^{f_{d_1}}| = |X^{f_{d_2}}|$.

Proof. There exists a bijection $\varphi : X^{f_{d_1}} \to X^{f_{d_2}}$, whereby any element $x \in X^{f_{d_1}}$ is mapped to an element in $X^{f_{d_2}}$ through multiplication by r_{90}. Similarly, an inverse function $\varphi^{-1} : X^{f_{d_2}} \to X^{f_{d_1}}$ exists, which maps every $z \in X^{f_{d_2}}$ to an element $x \in X^{f_{d_1}}$ through multiplication by ${r_{90}}^3$. □

There exists no systematic way to find all symmetric 5×5 Latin squares, other than by brute force. Thus, instead of working through all 161 280 squares, we shall instead just consider the 56 reduced squares ($R(5) = 56$). Through explicit analysis of a list of all 56 reduced 5×5 Latin squares, it is clear that there exist 6 symmetric reduced 5×5 Latin squares (squares 15, 24, 25, 39, 41, and 51, on this list).

Proposition 2.18. There exist 720 (not necessarily reduced) Latin squares symmetric about d_1, such that $X^g = 720$. That is,

$$X \cap \mathrm{Sym}_5 = 720.$$

Proof. As noted above, there exist 6 *reduced* 5×5 Latin squares which are symmetric about the diagonal d_1. There are 5! relabelings of any Latin square, and thus there are 5! ways to permute the labelling of these six reduced Latin squares such to make them non-reduced, but to maintain their symmetry about d_1. Hence, for each of the 6 symmetric reduced Latin squares, there are $5! = 120$ symmetric relabelings – so there are $6 \times 120 = 720$ symmetric (not necessarily reduced) Latin squares in X. □

With this, we can finally apply Burnside's Lemma (Theorem 1.8) to all 5×5 Latin squares, to find how many there are up to symmetry. To recall, the elements $r_{90}, {r_{90}}^2, {r_{90}}^3, f_v, f_h \in G$ each fix 0 elements of $X = \{5 \times 5 \text{ Latin Squares}\}$; $e \in G$ – the identity – fixes all 161 280 elements; and $f_{d_1}, f_{d_2} \in G$ fix 720 elements each respectively.

Thus, we have:

$$\begin{aligned} |X/G| &= \frac{1}{|G|} \sum_{g \in G} |X^g| \\ &= \frac{1}{8}(161\,280 + 720 + 720) \\ &= \frac{162\,720}{8} \\ &= 20\,340 \end{aligned} \tag{3}$$

In fact, generalising this to any odd $n \times n$ Latin square, with $X = \{n \times n \text{ Latin Squares}\}$, we have:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

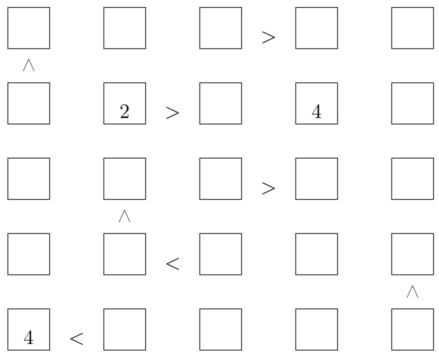

Figure 2.2: *An* instance *of a* 5×5 *Futoshiki puzzle with a standard set of clues.*

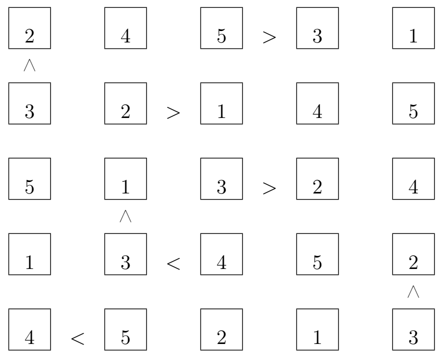

Figure 2.3: *The completed puzzle for the* instance *shown in Figure 2.2.*

$$= \frac{1}{8}\left(|X| + 2\left(n! \times |X \cap \mathrm{Sym}_n|\right)\right).$$

We can also carry out a similar process for n even, in which case $|X^{r_{90}{}^2}| \neq 0$ – so the rotation of 180° would indeed fix some Latin squares. With the standard-sized Futoshiki being 5×5, though, we are not interested in the case of n being even.

2.2 Futoshiki – Introducing the Inequalities

Futoshiki is a Japanese number puzzle made solvable by clues given in the form of strict inequality symbols ($<$ and $>$) which relate adjacent cells. Sometimes – depending on the number of inequality symbols present on the grid – there may also be some cells already filled in. A completed puzzle creates a Latin square. See Figures 2.2 and 2.3.

Definition 2.19. An *instance* is a configuration of inequality symbols relating a selection of adjacent cells in a partially-complete Latin square. The inequality symbols and pre-filled cells act as clues to help the solver complete the numbering of the grid.

2.2.1 Clues

In an $n \times n$ Futoshiki puzzle, there are $(n-1)$ spaces which may be occupied by an inequality symbol in every row and column respectively. With there being n rows and n columns, it follows that an $n \times n$ grid presents $n(n-1) + n(n-1) = 2(n^2 - n)$ spaces for inequality symbols to potentially be.

For each of the $2(n^2 - n)$ "clue spaces", each can take one of three statuses: it can have a $<$ symbol, a $>$ symbol, or it can remain blank. For the purpose of counting possible grids, though, we may simplify this to a binary choice.

Proposition 2.20. For each of the $2(n^2 - n)$ "clue spaces" in an $n \times n$ grid, there are two choices of its status: it can either have a clue in it, or it can remain blank. Thus, it follows that there are

$$2^{2(n^2-n)}$$

possible configurations of inequality symbols for every completed $n \times n$ Latin square.

Example 2.21. For each 5×5 Futoshiki grid ($n = 5$ – the typical size for this puzzle), there are

$$2^{2(20)} \approx 2^{1\,048\,576}$$

possible configurations for clues – ranging from no clues, to having clues between all adjacent cells (40 clues). With 20 340 unique grids up to symmetry (20 340 orbits), that makes $20\,340 \times 2^{1\,048\,576} \approx 2.2364 \times 10^{16}$ unique completed Futoshiki puzzles!

2.2.2 Symmetries

Like with Latin squares, we wish to count unique Futoshiki puzzles up to symmetry through the use of Burnside's Lemma (Theorem 1.8). As we saw with the Latin squares, the only symmetries $g \in D_4$ acting on X with any fixed points were e, f_{d_1}, f_{d_2} – so these are the only symmetries we shall consider for inequality symbols on the grid – as when we utilise Burnside's Lemma for the combination of Latin squares and configurations of inequality symbols, any symmetries with 0 fixed points for Latin squares will cancel out – as will be shown by Theorem 2.23.

Definition 2.22. Let us define the following sets:

- $X = \{5 \times 5 \text{ Latin squares}\}$,
- $Y = \{\text{Configurations of inequality symbols in a } 5 \times 5 \text{ Futoshiki grid}\}$,
- $Y' = \{\text{Configurations of positions for symbols in a } 5 \times 5 \text{ Futoshiki grid}\}$,
- $F = (X \times Y) = \{5 \times 5 \text{ Latin squares with configurations of inequality symbols overlaid}\}$,
- $F' = (X \times Y') = \{5 \times 5 \text{ Latin squares with configurations of positions for symbols overlaid}\}$.

It is trivial that the identity e fixes all 2.2364×10^{16} Futoshiki puzzles, but it is less trivial when considering the fixed points of $f_{d_1}, f_{d_2} \in G$. Like with Latin squares, the elements $y \in Y'$ fixed by f_{d_1} are the ones which are symmetric about the diagonal axis from the top-left to the bottom-right of the grid. When inserting inequality signs on grids such that they satisfy this, we are essentially filling-in two positions for each choice we make – so there are half as many choices to make – as every choice fills-in two positions. Thus, we only need to consider the filling-in of the positions on the bottom/left-side of our diagonal axis d_1 – as the positions on the top/right-side mirror the ones on the bottom/left. Hence, we have that there are $2^{\frac{1}{2}2(n^2-n)} = 2^{(n^2-n)}$ configurations of inequality signs which are symmetric about d_1. It follows from this that $|(Y')^{f_{d_1}}| = 2^{(n^2-n)} = 2^{20}$ for a 5×5 grid.

Similarly to with Latin squares, we can apply Lemma 2.17, which tells us that

$$|(Y')^{f_{d_1}}| = |(Y')^{f_{d_2}}|.$$

Theorem 2.23. For the group G acting on the sets X and Y' by the natural action of D_4 on the squares induced by X and Y' respectively, as defined in Definition 2.22, we have

$$|(F')^g| = |X^g| \cdot |(Y')^g|.$$

Proof. Let $x \in X$ and $y \in Y'$ – such that $(x, y) \in (F')^g \subset F'$. Also, let $g \in D_4$.

$$\begin{aligned} (x,y) \in (F')^g &\iff g \cdot (x,y) = (x,y) \\ &\iff (g \cdot x, g \cdot y) = (x,y) \\ &\iff \begin{cases} g \cdot x = x, \\ g \cdot y = y. \end{cases} \\ &\iff \begin{cases} x \in X^g, \\ y \in (Y')^g. \end{cases} \end{aligned}$$

Hence, all ordered pairs (x, y) are in $(F')^g$ precisely when both $x \in X^g$ and $y \in (Y')^g$. So, for every $x \in X^g$, any element $y \in (Y')^g$ can be overlaid to create an element of $X \times Y'$ which is fixed by g – so there are $|(Y')^g|$ fixed overlay configurations for every $x \in X^g$ – thus, $|(F')^g| = |X^g| \cdot |(Y')^g|$. □

Whilst Theorem 2.23 is profound in working toward the eventuality of counting the total number of Futoshiki puzzles up to symmetry by Burnside's lemma, the sets Y' and F' do not relate directly to Futoshiki puzzles. We shall thus show that there is a D_4-*equivariant* bijection between the D_4-sets F' and F (and consequently, one between Y' and Y too).

Definition 2.24. A $G-$*set* is a set S together with an action of a group G.

Definition 2.25. A map $f : X \to Y$ between two $G-$sets X, Y is $G-$*equivariant* if

$$f(g \cdot x) = g \cdot (f(x)),$$

for all $g \in G$ and $x \in X$.

First, we shall recognise a bijection φ between F' and F, before going on to show that it is D_4-equivariant.

Proposition 2.26. There exists a bijection $\varphi : F' \to F$.

Proof. We shall prove the bijectivity of a function $\varphi : F' \to F$ through first defining it and proving its existence as an injective function, and then defining a valid inverse function $\psi : F \to F'$, to prove surjectivity.

Given $(x, y') \in F'$ and $(x, y) \in F$, we may define $\varphi : F' \to F$ to send $(x, y') \mapsto (x, y) \in F$ such that the inequality symbols in $y \in Y$ take the same configuration as the places for symbols in $y' \in Y'$. As $(x, y) \in F$ is a valid Futoshiki puzzle, there is only one valid y which can be mapped to by $\varphi(y')$ – as there is only one valid configuration of inequality symbols in the layout presented in y' which creates a valid Futoshiki puzzle from $x \in X$. Hence, φ is an injection from F' to F.

Next, we shall consider a map $\psi : F \to F'$, which sends $(x, y) \mapsto (x, y')$. This shall leave $x \in X$ fixed, but send the configuration of symbols $y \in Y$ to the anonymised configuration of symbol places $y \in Y'$. Every $y' \in Y$ maps to precisely one $y' \in Y$ by this function. In fact, it shall send

$\varphi(y')$ to y' – and so $\psi : F \to F'$ defines an inverse of $\varphi : F' \to F$; thus proving subjectivity and consequently, bijectivity.

Hence, we have shown that there exists a bijection $\varphi : F' \to F$. □

Lemma 2.27. There exists a D_4-equivariant bijection $\varphi : F' \to F$.

Proof. As defined in the proof of Proposition 2.26, there exists a bijection $\phi : F' \to F$. Now, it suffices that the conditions of Definition 2.25 are satisfied by such a bijection. For all $(x, y') \in F'$,

$$\begin{aligned} g \cdot \varphi((x, y')) = \varphi(g \cdot (x, y')) &\iff g \cdot (x, y) = \varphi(g \cdot x, g \cdot y') \\ &\iff (g \cdot x, g \cdot y) = (g \cdot x, g \cdot y). \end{aligned}$$

Hence, the bijection defined in Proposition 2.26 is equivariant. □

Corollary 2.28. $|F^g| = |(F')^g| = |X^g| \cdot |(Y')^g|$, as a result of Theorem 2.23 and Lemma 2.27.

We can now go on to find the total number of completed 5×5 Futoshiki grids by the use of Burnside's Lemma (Theorem 1.8) and Corollary 2.28 (and the results leading up to it). We have:

$$\begin{aligned} |F/G| &= \frac{1}{|G|} \sum_{g \in G} |F^g| \\ &= \frac{1}{8} \sum_{g \in G} |X^g| \cdot |(Y')^g| \\ &= \frac{1}{8} \left(161\,280 \cdot 2^{2(5^2-5)} + 720 \cdot 2^{5^2-5} + 720 \cdot 2^{5^2-5}\right) \\ &= 22\,166\,154\,604\,707\,840 \text{ Futoshiki grids of dimension } 5 \times 5. \end{aligned} \tag{4}$$

2.3 Minimum Number of Clues Required for a Unique Futoshiki Puzzle

For a Futoshiki puzzle to be uniquely solvable, there must be a minimum amount of clues. For example, for a 5×5 grid with no inequality symbols on it, any of the 161 280 5×5 Latin squares is a valid and correct solution.

To be Completed...

※ Cross-Sum / Cross-Number

A cross sum is a puzzle based around a 3×3 square which is to be filled with the numbers $1 - 9$, such to satisfy a set of equations. Clues are given as $+$, $-$, $\times$, and $\div$ symbols between adjacently neighbouring cells and a positive integer at the end of each row and column. The solver must insert numbers into the square such to satisfy the equations in each row and column which are set out by the arithmetical operators between cells and the integer at the end of each row and column, which that row and column should be equal to.

$$
\begin{array}{ccccccc}
\boxed{} & + & \boxed{} & \times & \boxed{} & = & 70 \\
- & & + & & + & & \\
\boxed{} & - & \boxed{} & \times & \boxed{} & = & 8 \\
- & & \div & & + & & \\
\boxed{} & + & \boxed{} & \times & \boxed{} & = & 81 \\
\parallel & & \parallel & & \parallel & & \\
3 & & 1 & & 18 & &
\end{array}
$$

Figure 3.1: *A Cross-Sum puzzle, with clues as of arithmetical operators and row/column totals.*

$$
\begin{array}{ccccccc}
\boxed{8} & + & \boxed{6} & \times & \boxed{5} & = & 70 \\
- & & + & & + & & \\
\boxed{3} & - & \boxed{1} & \times & \boxed{4} & = & 8 \\
- & & \div & & + & & \\
\boxed{2} & + & \boxed{7} & \times & \boxed{9} & = & 81 \\
\parallel & & \parallel & & \parallel & & \\
3 & & 1 & & 18 & &
\end{array}
$$

Figure 3.2: *The solution for the puzzle shown in Figure 3.1.*

It is important to note that cross-sum puzzles do not work by the typical order of operations (BIDMAS). Instead, the puzzle works from left-to-right in rows and top-to-bottom in columns. For example, in the solution shown in Figure 3.2, we see that the central column reads

$$6 + 1 \div 7 = 1.$$

Of course, according to the order of operations in mathematics, this is not true, and in fact, the left-hand side would read $6 + \frac{1}{7}$, which is equal to $\frac{43}{7}$ – not 1.

In cross-sum, working from top-to-bottom, we can interpret this column to say

$$(6 + 1) \div 7 = 1,$$

which, of course, is true – as $(6 + 1) = 7$, and $7 \div 7 = 1$.

Another notable feature of the puzzle is that you never leave the set of natural numbers $\mathbb{N}$. For example, if a row of a puzzle was

$$\boxed{} \quad \div \quad \boxed{} \quad \times \quad \boxed{} \quad = \quad 12,$$

although we have a valid solution given by

$$\boxed{3} \quad \div \quad \boxed{2} \quad \times \quad \boxed{8} \quad = \quad 12,$$

we have that $3 \div 2 = 1.5 \notin \mathbb{N}$. Thus, as this does not stay within the natural numbers when working left-to-right, the correct solution to this row would in fact be

$$\boxed{6} \quad \div \quad \boxed{2} \quad \times \quad \boxed{4} \quad = \quad 12.$$

3.1 Enumeration

In enumerating cross-sum puzzles, we shall first consider an upper-bound for the number of fully-filled grids which exist – before considering how many *valid* grids do exist. We shall begin by noting that there are 9 cells to be filled by the numbers $1-9$, each appearing exactly once, and there are 12 places for operation symbols $(+, -, \times, \div)$ to be placed.

Lemma 3.1. Through considering permutations of the numbers $1-9$, there are $9! = 362\,880$ arrangements of the numbers into the cells on the grid – all of which create at least 4096 valid grids.

Proof. It is trivial that there are $9! = 362\,880$ arrangements of the numbers into the cells on the grid, so we shall just prove that all of these create at least one valid grids.

It is a fact that the natural numbers are closed under addition and multiplication. Hence, any grid with a valid arrangement of the 9 numbers and only containing the operators $+$ and $\times$ will be valid. For each of the 362 880 grids, there are 12 places for operation symbols and each of these can be either $+$ or $\times$. Hence, there is a binary choice of operators for each of the 12 places – so there are $2^1 2 = 4096$ operator configurations only involving these two operators. □

Lemma 3.2. There exists 16 777 216 configurations of operators on the cross-sum grid.

Proof. There are 12 places for operators on the grid, and there are 4 choices of operator per grid. Therefore, by elementary combinatorics, there are $4^1 2 = 16\,777\,216$ configurations of operators on the cross-sum grid. □

Neglecting the fact that the puzzle works within the set of natural numbers, we shall now consider an upper bound for how many possible grids there are.

Proposition 3.3. 6 088 116 142 080 is an upper bound for the number of valid cross-sum grids.

Proof. By Lemma 3.1, there are 362 880 configurations of the numbers on the grid, and by Lemma 3.2, there are 16 777 216 possible configurations of the operator symbols on the grid.

For each configuration of numbers, we can overlay every configuration of operator symbols. Hence, there are

$$362\,880 \times 16\,777\,216 = 6\,088\,116\,142\,080$$

possible combinations of number configurations and operator configurations. □

Not all combinations of number layouts and operator symbol layouts make valid puzzles, though – hence why this is an upper bound. In fact, we can actually get a more accurate upper bound by considering the behaviour of certain operators with our set of numbers $1-9$.

Lemma 3.4. In a puzzle constructed using only the $-$ operator, the numbers 1, 2 and 3 cannot be present in the first row or column.

Proof. If the first cell of a row or column only using the $-$ operator was 3, then to stay in $\mathbb{N}$, we must only subtract numbers less than 3. Hence, the second and third cells could only take the values 1 and 2, but $3 - 1 - 2 = 0 \notin \mathbb{N}$. Thus, 3 cannot be in the first cell.

If a 2 was present in the first row or column, then we would have a row or column which reads

$$\boxed{2} \quad - \quad \boxed{} \quad - \quad \boxed{} \quad = \quad n.$$

It is trivially impossible for n to be a natural number for any enumeration of the second and third number boxes – as the smallest numbers which can be validly placed in these boxes are 1 and 3 – which makes $n = -2 \notin \mathbb{N}$.

Similarly, if 1 was in the first box, the same argument works. $\square$

Lemma 3.5. In a puzzle constructed using only the $\div$ operator, no prime number can be present in the first row or column.

Proof. A prime number p has no positive divisors other than itself and 1. This means, if p is in the first cell of a row or a column of a puzzle constructed using only the $\div$ operator, the only valid entry for the second cell is 1.

$$\boxed{p} \quad \div \quad \boxed{1} \quad \div \quad \boxed{} \quad = \quad n.$$

Of course, this means that the first two cells equate to p – and so in order for n to be a natural number, the third cell must equal a positive divisor of p – but p is prime and its only divisors are itself and 1, both of which are already present in the row/column. Hence, there is no valid completion of a row or column of a puzzle using only the $\div$ operator if the first cell of that row/column is occupied by a prime number. $\square$

Corollary 3.6. In a puzzle using only the $\div$ operator, the number 1 cannot be in the first cell of any row or column.

Proof. 1 has no divisors in $\mathbb{N}$ other than itself, and so it is immediately obvious that 1 cannot be inserted into the first cell of a row or column in a cross-sum, as the only entry which could be inserted into the second cell is 1, but the same number cannot appear twice in the puzzle. $\square$

Lemma 3.7. The top-left cell in a puzzle using only the $-$ operator cannot be $1, 2, 3, 4, 5, 6, 7,$ or 8.

Proof. From Lemma 3.4, we know that the numbers $1, 2, 3$ are invalid entries for any cells in the first row or column, and we also know by definition of the puzzle that each number $1 - 9$ can only appear once in a grid. Suppose the top-left entry $x \neq 9$ – then $3 < x < 9$. The top row of the puzzle is

$$\boxed{x} \quad - \quad \boxed{} \quad - \quad \boxed{} \quad = \quad n,$$

for some $n \in \mathbb{N}$. To avoid creating an invalid row, we must avoid the presence of the numbers $1, 2, 3$ – thus, the second cell cannot be occupied by $1, 2, 3, (x-4), (x-3), (x-2), (x-1)$, as any of these would cause the third cell to be 0, 1, 2, or 3 (of course, 0 is not valid either). For $x = 8$, this means that the second cell cannot be occupied by $1, 2, 3, 4, 5, 6, 7$ – leaving no valid entries for this cell. Similarly, for $x = 7, 6, 5, 4, 3, 2, 1$, we yield the same finding. $\square$

Proposition 3.8. There are no valid puzzles using only the $-$ operator.

Proof. By Lemma 3.7, we see that the top-left cell cannot be equal to any number less than 9 – so let it equal 9. Then, we have that the top row is

$$\boxed{9} - \boxed{} - \boxed{} = n,$$

We have that the second cell must not equal $1, 2, 3, 5, 6, 7, 8$ (by the proof of Lemma 3.7) – so there is a valid entry for the second cell:

$$\boxed{9} - \boxed{4} - \boxed{} = n.$$

However, in order for n to be a natural number, the third cell must be equal to 1, 2, 3, or 4 – all of which are invalid entries. Thus, there are no valid completions of the first row/column of a Cross-sum puzzle which uses only the $-$ operator. □

Proposition 3.9. There are no valid puzzles using only the $\div$ operator.

Proof. As a result of Lemma 3.5 and Corollary 3.6, we know that the numbers $1, 2, 3, 5, 7$ cannot occupy a cell within the first row or column – leaving only the numbers $4, 6, 8, 9$ valid for the cells of the first row and column. However, there are 5 cells which are in the union of the first row and column, but only four valid entries for them. Thus, by the pigeonhole principle, there does not exist a valid puzzle using only the $\div$ operator. □

Proposition 3.10. There are no valid puzzles using only a combination of the $-$ and $\div$ operators.

Proof. □

To be Completed...

※ GAP Code

The following code can be used in GAP to determine that the two Shidoku grids on page 5 are essentially different and thus in different orbits.

```
# Written by Carl-Fredrik Nyberg-Brodda

AreEssentiallyDifferent := function(x, y)
    local k, H, G, K;

    H := RowColumnSymmetryGroup(); # 128
    G := SymmetricGroup(4); # 24

    K := DirectProduct(G, H); # 3072


    for k in Elements(K) do
        if action_direct(x, k)=y then
            Print("The square \n\n");
            PrettyPrint(x);
            Print("\n\nand the square \n\n");
            PrettyPrint(y);
            Print("\n\nare not essentially different; they are related by the element ", k, "\n");
            return false;
        fi;
    od;
    Print("The square \n\n");
            PrettyPrint(x);
            Print("\n\nand the square \n\n");
            PrettyPrint(y);
            Print("\n\nARE essentially different.\n");
    return true;

end;

# Returns the square as a list [1..n^2], read along the rows.
SquareAsList := function(S)
    local l, i, j;
    l := [];
    for i in [1..Length(S)] do
        for j in [1..Length(S)] do
            Add(l, S[i][j]);
        od;
    od;
    return l;
end;

# Inverse of the above. Takes parameter n for convenience.
ListAsSquare := function(l, n)
    local T, i, j;
    T := [];
```

```
    for i in [1..n] do
        T[i] := [];
        for j in [1..n] do
            T[i][j] := l[n*(i-1)+j];
        od;
    od;
    return T;

end;



## D is the set of Shidokus in matrix form
## S is the set of Shidokus in list form
MakeShidokus := function()
    local A, C, D, i, S;
    A:=Arrangements([1,2,3,4],4);;
    C:=Cartesian(A,A);;
    C:=Filtered(C,c->not c[1][1]=c[2][1] and not c[1][2]=c[2][2] and not c[1][3]=
        c[2][3] and not c[1][4]=c[2][4]);;
    C:=Filtered(C,c->Set([c[1][1],c[1][2],c[2][1],c[2][2]])=[1,2,3,4]);;
    D:=Cartesian(C,C);;
    D:=Filtered(D,d->Set([d[1][1][1],d[1][2][1],d[2][1][1],d[2][2][1]])
        =[1,2,3,4]);;
    D:=Filtered(D,d->Set([d[1][1][3],d[1][2][3],d[2][1][3],d[2][2][3]])
        =[1,2,3,4]);;
    for i in [1..Length(D)] do
        D[i]:=[D[i][1][1],D[i][1][2],D[i][2][1],D[i][2][2]];
    od;
    S:=[];
    for i in [1..Length(D)] do
        S[i]:=Flat(D[i]);
    od;

    return D;
end;




RowColumnSymmetryGroup := function()
    local g1, g2, g3, g4, g5, g6, g7, g8, g9, G;
    g1:=(1,5)(2,6)(3,7)(4,8);; #Swaps rows 1 and 2
    g2:=(9,13)(10,14)(11,15)(12,16);; #Swaps rows 3 and 4
    g3:=(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16);; #Swaps horizontal bands
    g4:=(1,2)(5,6)(9,10)(13,14);; #Swaps cols 1 and 2
    g5:=(3,4)(7,8)(11,12)(15,16);; #Swaps cols 3 and 4
    g6:=(1,3)(5,7)(9,11)(13,15)(2,4)(6,8)(10,12)(14,16);; #Swaps vertical bands
    g7:=(2,5)(3,9)(4,13)(7,10)(8,14)(12,15);; #Reflection in diagonal
    g8:=(3,8)(2,12)(1,16)(6,11)(5,15)(9,14);; #Reflection in diagonal
    g9:=(1,13,16,4)(2,9,15,8)(3,5,14,12)(6,10,11,7);; #Rotation by 90
    G:=Group(g1,g2,g3,g4,g5,g6,g7,g8,g9);;
    return G;
```

```
end;


action := function(S, h)
    local l, i, new_l;
    l := SquareAsList(S);
    new_l := [];
    for i in [1..Length(l)] do
        new_l[i] := ShallowCopy(l[i^h]);
    od;
    return ListAsSquare(new_l, Length(S));

end;


action_swap_letters := function(S, g)
    local l, i, new_l;
    l := SquareAsList(S);
    new_l := [];
    for i in [1..Length(l)] do
        new_l[i] := ShallowCopy(l[i]^g);
    od;
    return ListAsSquare(new_l, Length(S));

end;

action_direct := function(S, h)
    local l, i, new_l;
    l := SquareAsList(S);
    new_l := [];
    for i in [1..Length(l)] do
        new_l[i] := ShallowCopy((l[(i+4)^h-4])^h);
    od;
    return ListAsSquare(new_l, Length(S));

end;



# Checks if the Shidoku x is a fixed point of the group element h
IsFixedPoint := function(x, h)
    return action_direct(x, h)=x;
end;

#
FixedPoints := function(Y, h)
    return Filtered(Y, y->IsFixedPoint(y, h));
end;


BurnsideLemma := function(Y, H)
    local avg, h, fp;
    avg := 0;
```

```
    for h in H do
        fp := FixedPoints(Y, h);
        if Length(fp)>0 then
            Print("The element ", h, " fixes ", Length(fp), " Shidoku grids.\n");
            avg := avg+Length(fp);
        fi;
    od;
    Print("Hence: the average number of fixed points is ", avg/Order(H), "\n");
    #return avg/Order(H);
end;


# Prints the grid in a nice way
PrettyPrint := function(x)
    local i, j;
    for i in [1..Length(x)] do
        for j in [1..Length(x)] do
            Print(x[i][j], " ");
        od;
        if i<>Length(x) then Print("\n"); fi;
    od;
end;
```

Defining x and y as the Shidoku grids we wish to compare,

```
        x:=[[1,2,3,4],[3,4,1,2],[2,1,4,3],[4,3,2,1]];
        y:=[[1,2,3,4],[3,4,1,2],[2,3,4,1],[4,1,2,3]];
```

the command `AreEssentiallyDifferent(x,y);` can be used to return the following output:

```
gap>    AreEssentiallyDifferent(x,y);
        The square

        1 2 3 4
        3 4 1 2
        2 1 4 3
        4 3 2 1

        and the square

        1 2 3 4
        3 4 1 2
        2 3 4 1
        4 1 2 3

        ARE essentially different.
        true
```

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