MTHA6005Y Mathematics Project 2020/21

Popular Mathematics

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Introduction

"*To not know mathematics is a severe limitation to understanding the world.*"

\- Feynman (1999), *The Pleasure of Finding Things Out.*

Mathematics is an underlying element of everything around us and so it comes as no surprise that in an increasingly globalising world, becoming steadily more interconnected, exposure to mathematics is becoming more and more common. Just in the past year, as a result of COVID-19, the general public has been exposed to mathematics every day, with graphs and statistics playing a crucial role in our understanding of the situation.

With this increased exposure to mathematics, interest in topics of mathematics is becoming increasingly common among the general public. However, The Department of Business, Innovation and Skills (2012) reports from a 2011 survey that 53% of the UK population aged 16–65 have numeracy levels equivalent to entry level 2 (GCSE level) or below, so it is clear that most people don't possess the mathematical skills to be able to fully understand the mathematics they may have developed interest in. Moreover, Chinn (2012) states that "maths makes some people feel anxious, leading them to avoid situations where they may have to use mathematics".

Strogatz (2014) recognises that mathematics becomes extremely tricky and "ethereal" to the popular mind when we introduce formulas, functions, theorems, and proofs. He indicates that most do not know how to pronounce common symbols in mathematics, such as $\oint$, and so it is ineffective to use such symbols when explaining mathematics to the common person, to whom these symbols seem alien. As such, one may ask:

> *How can the common mind pursue an interest in mathematics despite having little understanding or confidence when it comes to mathematics?*

Popular mathematics books are precisely the answer to this question. These take complicated or advanced mathematical topics and repackage them such to be conveyed in a way that can be understood by almost anyone, making mathematics accessible by most people – even those who may otherwise struggle to understand it.

In this document, I will explore the techniques used by authors of popular mathematics books, which collectively work to make advanced topics in mathematics seem simple and relatable to by the common mind (section 1). I will go on to apply these techniques in creating my own popular mathematics article involving Sudoku puzzles (section 2), before comparing the style and techniques used in my article to those of a formal mathematical text (section 3).

1 Techniques Used in Popular Mathematics Publications

"*Mathematics is a language plus reasoning; it is like a language plus logic. Mathematics is a tool for reasoning.*"

- Feynman (1965), *The Character of Physical Law.*

Across the whole genre of *Popular Science*, there appear to be many common themes and techniques portrayed and utilised – from building a rapport with the reader and adopting a casual, friendly tone of language (as described by Strogatz (2014)), to using anecdotes and analogies, which the reader can almost certainly relate to, in order to help them understand advanced topics in mathematics – as demonstrated by Hayes (2008).

Illuminate

Mathematics can be daunting, and thus it can be easy to give up with if one feels that they are not getting anywhere with it. Strogatz (2014) therefore alludes to the importance of *illuminating* the reader – providing them with "a-ha!" moments of revelation throughout the text, in order to make them feel like they are achieving a good understanding of what they are reading – motivating them to read on. Furthermore, Strogatz (2014) claims that moments of illumination help the reader see the beauty of mathematics and fall in love with that – especially after being clueless for so long.

Use Numerical Examples Instead of Generalised Cases

Generalisations of equations, using letters and symbols, can cultivate confusion, as noted by Cheng (2015). For the untrained mind, this abstraction is not natural to be able to follow, and often in order to understand such an abstraction, one usually constructs some examples of their own. Due to the nature and target readers of popular mathematics books, it may be deemed unnecessary, over-complicating and demoralising to use generalised cases and derive general formulae. It could be more effective to simply work through a specific example. For instance, when writing about the parabolic motion of a football, instead of considering all possible paths, just go into depth with one – maybe, that which takes the ball from the centre-line of the pitch to the back of the net.

Make Connections to the Real World – Teach Maths as a Humanised Subject

Analogies and real-world links can be extremely effective in both attracting a reader's interest into a mathematical subject, and subsequently simplifying advanced mathematical topics to be easier to understand for the common mind. For example, Elwes (2011) has chapters entitled *How to make a million on the stock market*, and *How to build the perfect beehive*; obviously drawing on real world applications of the maths described within, in order to entice the reader – and Hayes (2008) uses the analogy of flipping a mattress to describe symmetries in Group Theory.

Palisoc (2014) indicates that mathematics should be taught as a language and through analogies, for – to the untrained mind – sophisticated symbols and equations can be daunting and nonsensical. Additionally, Strogatz (2014) suggests that algebraic manipulations should be kept to a minimum, and we should instead represent mathematical ideas pictorially. It would therefore be ineffective to popularise mechanics, for example, algebraically and through generic formulae and no applied or numerical examples. Instead, it could be better to do this through the running example of a game of snooker. The reader can picture a snooker table, but they may struggle to even understand what is meant by "particles travelling through the Euclidean space".

Treat the Reader as a Friend

Strogatz (2014) identifies that explaining mathematics requires empathy by both parties – the author and the reader. The author must empathise with the reader, in order to correctly present mathematical topics in a way which can be appropriately and effectively understood by the reader, and the reader must be able to empathise with the author and their words, in order to not be left feeling confused or alone. Applying this observation, Strogatz (2012) – and many other authors – employ many techniques to incite a rapport between themselves (the authors) and the readers, such as:

- Addressing the reader directly, as demonstrated by Feynman (1965), using phrases like "I gave you the equation just to impress you with the speed with which mathematical symbols can convey information", and "you might want to look further". Feynman also uses rhetorical questions in order to make the reader feel involved and like they're being directly spoken to – making them feel more engaged and immersed in the topic he is discussing.
- Use of informal, friendly language and casual tone – or anything which escapes the rigid formality of a textbook, which may deter the reader's interest. Cheng (2015) gives a good example of this, with the use of contractions and writing in prose: talking around the point being made – rather than giving just the key information and expecting the reader to extrapolate the rest, as is typical in a textbook.
- Anecdotal segues into mathematical topics. For example, again when discussing abstraction in mathematics, Cheng (2015) begins by talking about tidying her

kitchen and "putting away the equipment and ingredients that you don't need", which segues into the role of abstraction in mathematics.

- A reminder that the author is human, and thus reference to other people – perhaps friends of the author who inspired the article. Strogatz (2012) begins by describing a friend with a deep interest in science, but very little understanding of it, and so Strogatz describes his science-based conversations with this friend to the reader as the inspiration for his writing. Enzensberger et al. (1998) provides an extreme example, conveying mathematical knowledge through an entire fictional story.

All of these collectively cultivate a casual, conversation-like feel to the text and thus echo the message that even if you're not a mathematically-minded person, you are always welcome in the world of mathematics.

Furthermore, Strogatz (2014) advises that when pictures and diagrams are used, they should be colourful and cartoon-like, to seem more inviting, and they should be placed in a casual way, such that the text flows around it. They should not be numbered, and captions should be integrated in the main body of the text, rather than directly below the figure, as this would give a formal textbook feel to the text.

Guide and Engage with the Reader's Thoughts

Whilst writing for someone for whom mathematics and a scientific way of thought may not be their strong point, it may be such that the reader may not know what questions to ask and when to ask them. It is thus beneficial to pose these questions rhetorically throughout the text, or to perhaps introduce a fictional skeptic, to whom the text is addressed, and the skeptic contributes through the input of questions and observations, which the author intends for the reader to consider.

A technique commonly adopted to engage the reader is the use of a dialogue – one which can be observed in Plato and Cornford (1941), and, in the genre of popular mathematics, it is used by Flannery (2006) and Knuth (1974). This comes from a key element of mathematics being the act of asking questions: *Why does x imply y? What does this result tell us? Where can we go next?*

The introduction of a third-person, or a dialogue, keeps the reader engaged and ushers their thoughts to where they would be best served, according to the author's intentions.

2 A Popular Write-Up involving Sudoku

> "*If you want someone to follow your mathematical disquisitions voluntarily – or better yet, happily – you have to help him or her love the questions you're asking.*"
>
> - Strogatz (2014), *Writing about Math for the Perplexed and the Traumatized.*

In this section, I will utilise some of the techniques elucidated in section 1, to produce a popular article about Sudoku puzzles – which most people are familiar with. In describing the process of enumerating a Sudoku grid, I hope to provide the reader with a basic understanding of Combinatorics and Group Theory.

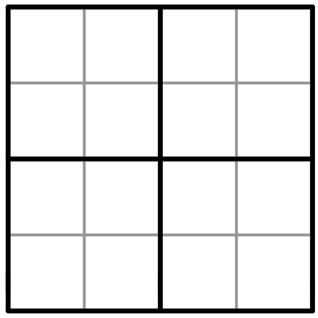
Why Sudoku Puzzles are made like a Cup of Tea

Over the past few months, my friend, Jess, and I have worked into our daily routines an hour or so of puzzle solving. Anything from word puzzles, picture puzzles, and many more. Of course, as a mathematician, I love number puzzles; Futoshiki, Kakuro, Cross Sums – you name it, I probably enjoy it. Jess isn't a huge fan of number puzzles, but the most famous of all number puzzles has to be Sudoku, and so occasionally I do get her to have a go at solving one. She's very slow at doing them and guesses a lot of things, leading to a trial-and-error style method of working to solve the puzzle. I, on the other hand, am very systematic in my method and I like to go around the entire grid, penciling in any possible numbers for each cell before even starting to fill in the solution. This way, I find the puzzle begins to solve itself after a little while.

Just last week, Jess and I were working through a Sudoku puzzle when she stopped and proposed a question:

How many possible Sudoku grids are there?

At first glance, I thought this quite simple question would have quite a simple solution, so I got my pencil out and started jotting down numbers. An hour later, I was still working on it and I simply couldn't get very far. I got online and looked it up, and – to my surprise – I saw that it's a question which had only been properly answered within the past 15 years. From this realisation, I began to think that maybe it was quite a complex question after all. You see, when mathematicians enumerate Sudoku grids, they don't just sit and count; they calculate various complicated invariants, linking together all kinds of differing mathematical topics. On further inspection though, I realised that this really isn't the case. In fact, if you know how to make a cup of tea, then you're on your way to answering Jess' question!

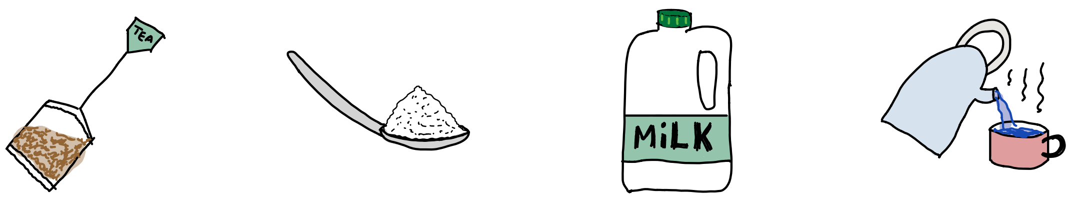


A typical Sudoku grid is a 9×9 grid, divided into 9 blocks containing 3×3 cells. For the sake of simplification though, let's look at a 2×2 grid of blocks containing 2×2 cells, as shown to the right. This Sudoku variation contains the numbers 1, 2, 3 and 4, once in each row, column and 2×2 block. All that we'll do with the 4×4 grid is equally valid with a full sized 9×9 grid though, so if you can find how many possible enumerations of a smaller grid there are, then you're more than capable of finding out that but of a larger grid! Of course, before going any further, I'll address what you're probably thinking from the title:

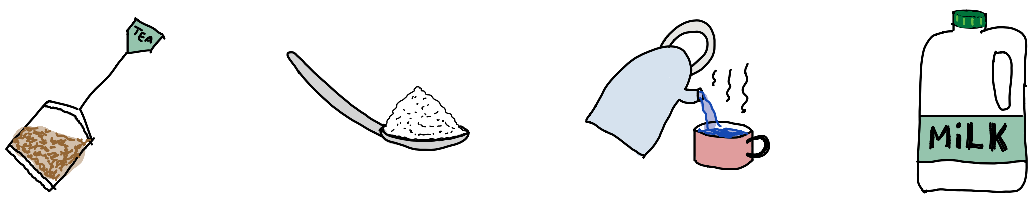
How on Earth does Sudoku relate to making a cup of tea?

Making a cuppa, you typically have four ingredients: the water, the teabag, the sugar (let's assume you have sugar), and the milk, and – although some may take offence – you have a choice of what order to put these things to the mug. I like to start with the teabag – as do most people – before adding the sugar, then the milk, and finishing off with the hot water, before letting it brew for two minutes. Some people questionably start with the sugar though, and – dare I say it – some even put the hot water in first.

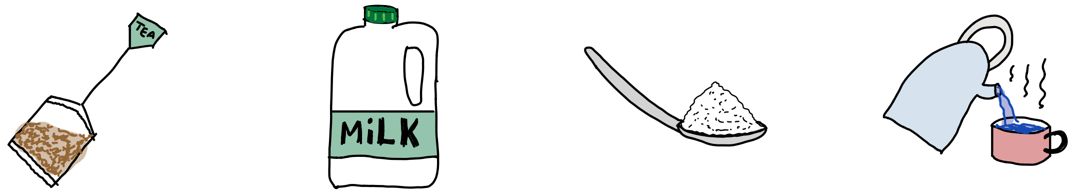
You are probably still asking why this is relevant. Well, it comes down to the fact that both the cup of tea and each 2×2 block of the Sudoku grid are made up of four things which can be arranged in a whole load of different ways. For example, as I have said, I like to put the teabag in first, and then the sugar. Once I've done that, I have two possible ways to finish making the tea. Method one (my preferred method):

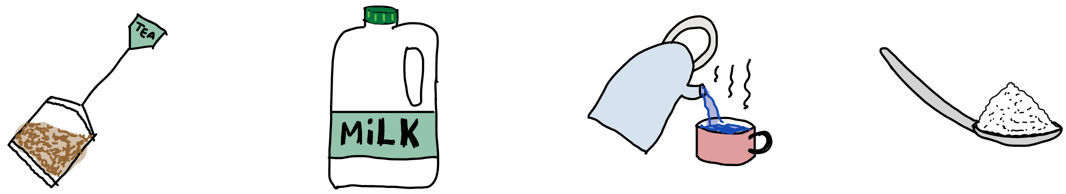


and – the quite questionable – method 2:



But what if I put the teabag in first still, but the milk in second? Then I'd have another two methods: methods 3 and 4 – shown below:





Similarly, if I were to keep the teabag first but add the hot water second, there would be two more methods created: one with the milk third and the sugar last, and one with the sugar third and the milk last.

These are all the possible orders – or *permutations*, as a mathematician would call them – of the four ingredients to make a cup of tea which involve adding the teabag first. *Permutation* is just a fancy way of referring to the number of ways you can arrange a set of objects or numbers. From this, we could label each of the ingredients of our cup of tea with a number between 1 and 4, and then think of the 2×2 grid almost as a storyboard. Let's denote each of the tea's ingredients as follows:

1	Tea Bag
2	Sugar
3	Milk
4	Hot Water

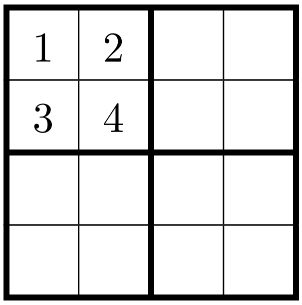
So, by this labelling, for someone who puts the teabag in first, then the sugar, before putting in the hot water and finishing off with the milk (method 2, above), their method is represented by the 2×2 "storyboard" grid:

1	2
4	3

What's this got to do with the Sudoku puzzle, though?

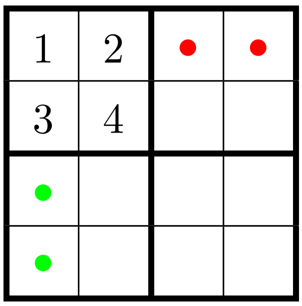
Well actually, it has everything to do with the Sudoku puzzle – and you may have already clocked why! Imagine this: you've boiled the kettle, you've got your teabag in hand, you've got your teaspoon of sugar ready to go, you've got the milk out of the fridge, and you've got your empty mug. How many different ways can we make the tea?

Well to start off with, we have a choice of four ingredients to put in first – four ways to start off our cup of tea: we could put the teabag in first, the sugar, the milk or – as painful as it is to say – we could even start with the hot water. For me, it's the teabag first, but choose whichever seems right to you. Up next, we have a choice of the three remaining ingredients, meaning there's three possible ways we can go. For me, it's either the sugar, the milk, or the hot water. Of course, I will choose the sugar. Next, we must decide which of the two remaining ingredients to put in first. For me, I have the choice of either the milk or the hot water, meaning there's two possible ways to finish off my cup of tea. I choose the milk, meaning that the only thing left to finish off with is the hot water.



So, there were 4 ways to choose how to start making the tea, then for each of those ways there was 3 ways to choose the second ingredient, followed by a further 2 ways to finish off making each of these cups of tea. From this, we can multiply 4 choices $\times$ 3 choices $\times$ 2 choices, which equals 24. This tells us that there are 24 permutations of the four ingredients. Similarly – and this is where it all comes together – there must be 24 possible permutations of the numbers $1, 2, 3$ and 4, just by representing each of the ingredients by one of the numbers, like before. This means there's 24 ways to enumerate one 2×2 block in the Sudoku grid.

All that work just for one block of the grid?



Now that we've established that there's 24 ways of enumerating the first block of our 4×4 grid, it's actually pretty simple to move on to the rest of the grid.

Looking at the first row of the grid on the left, there are two red dots in the third and fourth cells. We already have the numbers 1 and 2 in this row, so this means that the red dots must be replaced by the numbers 3 and 4 – but they can be either way around. Thus, there are two ways to complete the enumeration of row 1 – it can be either $1, 2, 3, 4$ or $1, 2, 4, 3$. In the very same way, we can complete the first column, so the green dots must be replaced by the numbers 2 and 4 (in any order).

Now, though, it gets a bit more complicated. We must move on to enumerating the bottom-right grid as completing row and column 2 in the same way that we did the first row and column would generate some invalid grids. Give it a go!

1	2	3	4
3	4		
2		a	b
4		c	d

A good observation to start with, is that there is a 4 in cell 3 or 4 of both the first row and the first column. This comes as a result of the 4 in our example being in the fourth position of the first block – meaning that the 4 in the first row and column cannot be in the first block, so must exist exactly once in red and exactly once in green. In fact, this will be the case for whatever number is in the fourth position of the first block (marked in blue), in all possible enumerations. This helps us out a lot, as it tells us that – in our example – the number 4 in the bottom right block must be in the place of the a; it cannot be in the fourth row as that row's 4 is already filled, and likewise it cannot be in the fourth column for the same reason.

You may also notice that there's a bit of restriction on where the other numbers can go in the bottom left block due to the enumeration of the first row and column respectively. In our example, there cannot be a 3 in position a or c, as there is already a 3 in the third column. Similarly, there cannot be a 2 in position a or b, as there is already a 2 in the third row. This means 3 can only be in position b or d, and 2 can only take position c or d – as, of course, 4 is in position a.

If we consider all possibilities from this observation, we have:

1	2	3	4
3	4		
2		4	3
4		2	1

1	2	3	4
3	4		
2		4	3
4		1	2

1	2	3	4
3	4		
2		4	1
4		2	3

So there are three possibilities to complete the bottom-right block. Furthermore, you may notice that there is now only one cell left to fill in each of the third and fourth rows and columns, and each of these only has one possibility – so these three completions of the bottom-right block are actually the only three completions of the whole grid!

But this is just for* your *example with the first block enumerated as* $1, 2, 3, 4$*.
How does it work for a different first block?

Whilst it may seem that way, it's actually quite simple to do this generally, which will tell us how many possible unique 4×4 Sudoku grids there are. We had $4 \times 3 \times 2 \times 1 = 24$ possible ways to enumerate the first block (4 choices for the first cell, times 3 for the second, and so on). In mathematics, we use the symbol "!" after a whole number to represent itself multiplied by all whole numbers less than it – we call it *factorial*. So, there are $4! = 4 \times 3 \times 2 \times 1 = 24$ (four factorial) ways to enumerate the first block.

Then, we said that – for every enumeration of the first block – there are two ways to finish off the first row and first column of the 4×4 grid. So for each of the 4! enumerations of the first block, there are 2! ways to finish enumerating the first row, and for each of the 4! enumerations of the first block and 2! enumerations of the first row, there are an additional 2! ways to finish the enumeration of the first column. From there, one cell of the bottom-right block will already be solved and you will encounter similar restrictions to those we saw above with the numbers 2 and 3 in the example, meaning that there are three valid enumerations of the bottom-right block for every enumerated first row and column. This, as above, completes the enumeration of the whole grid.

This means that in order to determine how many possible unique 4×4 Sudoku grids there are, we must multiply all of these numbers together, so we have:

$$
\begin{aligned}
\text{Possible Enumerations} &= 4! \times 2! \times 2! \times 3 \\
&= (4 \times 3 \times 2 \times 1) \times (2 \times 1) \times (2 \times 1) \times 3 \\
&= \mathbf{288}
\end{aligned}
$$

So... What next?

For some people, they could be satisfied with the answer of 288 and leave it there, but to other people they might define a "unique" enumeration slightly differently. For example, have a look at the following two enumerations. What do you notice?

1	2	3	4
3	4	1	2
2	3	4	1
4	1	2	3

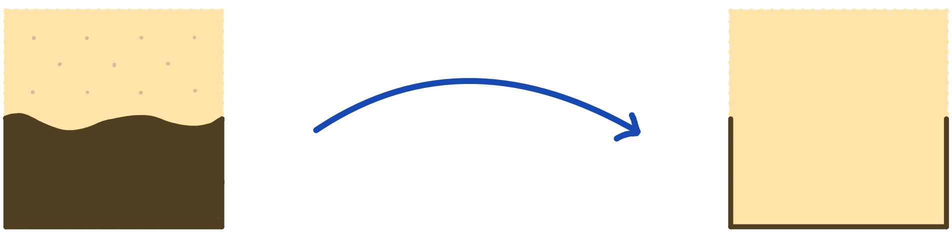
4	1	2	3
2	3	4	1
3	4	1	2
1	2	3	4

If you have a keen eye, you'll notice that the grid on the right has something in common with the one on the left: the cells have been flipped (or mirrored) over the horizontal centre-line – the numbers that were in the bottom row are now in the top row, and vice-versa. The top blocks in one grid are enumerated as the horizontal mirror image of the bottom blocks in the other grid, and the bottom blocks in one grid are enumerated as the horizontal mirror image of the top blocks from the other grid. From this, some may argue that these enumerations are therefore *"essentially the same"* and not unique.

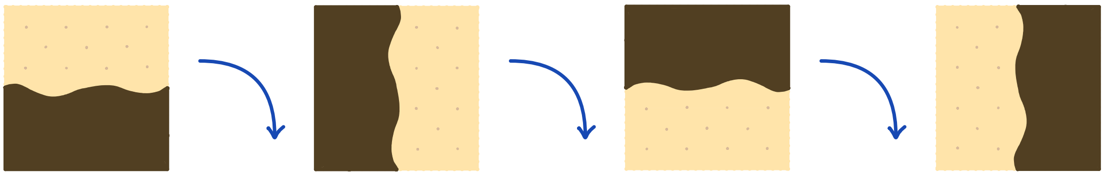
To further explore this, we can draw on an area of maths called *Group Theory*, in order to reduce our number 288 to factor in cases like the enumerations above – which are not really unique. On the surface of it, Group Theory is all about rotating and flipping shapes. In fact, it's actually very similar to dipping a biscuit in your tea.

Think of it this way: we've made our cup of tea (we've enumerated a Sudoku grid), and now it's time to enjoy it. We get out our biscuits – let's say they're squares of shortbread, half dipped in chocolate. I like to dip my biscuits straight down, so the down-facing edge is parallel with the surface of the tea when it begins to submerge, and so a flat face (either the front or back of the biscuit) is facing me. I find that this is the best way to prevent the disaster of it falling in the tea!

Group Theory allows us to find all possible ways of dunking the biscuit so that a flat edge is parallel with the surface of the tea. For example, it seems quite obvious to say that I could flip the biscuit 180°, so that the chocolate side faces away from me, but the chocolate half is still on the bottom part of the biscuit (As shown below). If I do this, my biscuit remains parallel to the surface of the tea – as I want it to be – and I have a face (not a thin edge) pointing at me. Great! If we did the same again, we'd get back to how we had the biscuit in the first place.

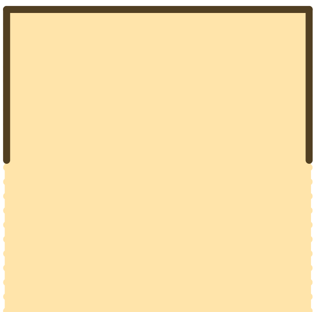


We can also rotate the biscuit by 90°. This way, we'd still have the front of the biscuit facing us, but we would have the chocolate part on the left. If we did this again, the chocolate would be on the top half, and again would bring the chocolate to the right side. Rotating one more time would put the chocolate back at the bottom. There are hence four ways of dunking the biscuit with the chocolate side facing us:



It follows that each of these rotations can also be flipped, as we did above with the original orientation of the biscuit. Group Theory tells us that as there are four possible rotations by 90° before we get back to the original orientation, the act of rotating by 90° has "*order* 4". Similarly, as the the biscuit returns to its original orientation after flipping it over horizontally twice, we can say that the act of flipping it horizontally has "*order* 2".

But what about flipping it vertically?

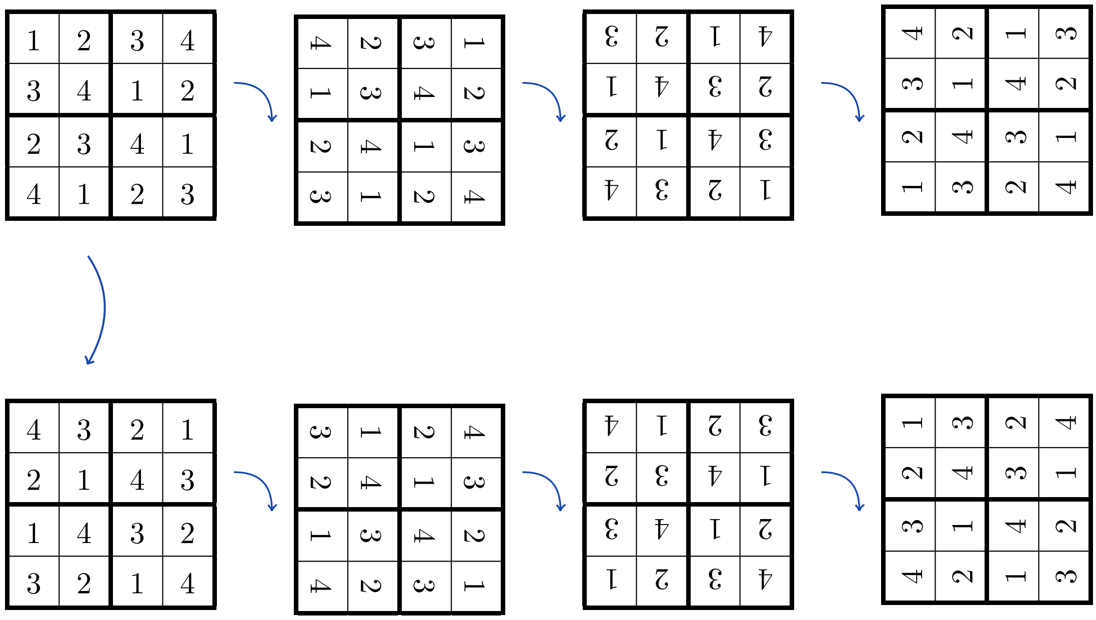


This is where Group Theory really comes in. If we flip the biscuit vertically, then it would look like the biscuit to the left. Group Theory tells us that this is actually equivalent to flipping the biscuit horizontally and then rotating it 90° twice. Thus, we've already factored in this orientation as it can be recreated using rotations and horizontal flips. In fact, we can recreate any orientation which is suitable for dunking through combinations of horizontal flips and rotations of 90°. In Group Theory, these orientations are called *symmetries.*

As the order of our horizontal flipping action is 2 and the order of the 90° rotation is 4, and there are no common symmetries generated by either of the actions of flipping or rotating alone, we can say that the biscuit therefore has 8 symmetries – because for each of the two sides of the biscuit, there are four orientations which make it suitable for dunking. Mathematicians call this specific collection of symmetries the *dihedral group* of order 8 – denoted D_4.

What's this got to do with Sudoku?

Just like our biscuit, each enumeration of the 4×4 Sudoku grid can be rotated four times and flipped horizontally for each rotation. It therefore also has 8 symmetries of this type (in most cases, anyway)! The total of 288 enumerations which we obtained earlier contains every one of these 8 symmetries for each enumeration. For example, taking the enumeration we had before, we can rotate it and flip it to get that it is *essentially the same* as 7 others:



(Of course, the numbers don't have to be sideways – that's just for illustration. They would be written the right way up in a real Sudoku.)

To factor for all of these *essentially the same* enumerations which are included in the total of 288, we must turn to an important result in Group Theory known as the *Orbit-Stabiliser Theorem* – more specifically, a variant of such called *Burnside's Lemma*.

Orbit? Stabiliser? Burnside? What are these?

William Burnside, a 19th Century English Mathematician, essentially founded modern Group Theory and quoted a result from German mathematician Frobenius, now known as *Burnside's Lemma*, in his 1897 book *Theory of Groups of Finite Order*, which gives a method of doing exactly what we want to do. In the context of Sudoku grids, it fundamentally states that, given an enumerated grid, the quantity of enumerated grids which can be reached through symmetries (the *orbit* of this enumeration) of this enumeration can be reduced to the number of *essentially different* grids by taking the sum of the number of grids fixed by any number of symmetries multiplied by the amount of symmetries which fixes them, and dividing by the total number of symmetries.

Some would argue that a truly *essentially different* enumeration is one which cannot be reached from another through rotating and flipping as we have as well as through the following symmetries additional to the ones we've already looked at:

- Swapping column 1 with 2 and/or 3 with 4.
- Swapping row 1 with 2 and/or 3 with 4.

- Swapping the first two columns with the second two columns.
- Swapping the first two rows with the second two rows.
- Relabelling entries (e.g. taking an enumeration and swapping all 2's with all 4's).
- Combinations of any of these.

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

If we take the enumeration to the right for example, we can apply the swapping of column 1 with 2 and 3 with 4, before relabelling the entries such that all 1's are swapped with all 2's, and all 3's are swapped with all 4's. This combination of permutations takes us right back to the grid on the right – as if nothing's happened. Therefore, this grid is fixed by this symmetry. In fact, it turns out that there are eight possible enumerations which have this same top-left block – these come from permuting columns 3 and 4 of rows 1 and 2, and permuting rows 3 and 4 of column 1, which in-turn changes the enumeration of the bottom-right block and rows 3 and 4 of column 2 (independently of each other) – generating $2 \times 2 \times 2 = 8$ different enumerations which are fixed by this symmetry.

If we consider all of the symmetries listed above and combinations of them, it turns out that there are 128 symmetries in total which have a fixed top-left block (and of course, we can make the first block anything we want through permutation of entries). Although the next step of counting orbits and stabilisers (symmetries which fix a grid) can be very cumbersome, mathematicians Elizabeth Arnold and Stephen Lucas used a computer to discover that of these 128 symmetries, 56 of them fix no 4×4 grids, 48 fix two, 9 fix four, 4 fix six, 6 fix eight, 4 fix ten, and 1 fixes all twelve. Utilising Burnside's Lemma, this gives:

$$\frac{(56 \times 0) + (48 \times 2) + (9 \times 4) + (4 \times 6) + (6 \times 8) + (4 \times 10) + (1 \times 12)}{128} = 2$$

which tells us that there are, in fact, only two *essentially different* enumerations, unique up-to symmetry. This can be quite easily observed in the three grids we found on page 9 – simply fill in the blanks to complete the grid, then it's quite straightforward to show that the second two grids are *essentially the same* by a horizontal flip, a 90° counter-clockwise rotation, and relabelling entries such that 2 and 3 are swapped. And that's it – there are just 2 *essentially different* enumerations of the 4×4 Sudoku grid!

As mentioned before, these very same methods can be used to find how many unique solutions there are to the standard 9×9 Sudoku grid – albeit with a few extra steps and some added complexity. For example, instead of enumerating based on a cup of tea having four ingredients, you'd need to consider nine ingredients. Maybe it's one of those fancy coffee-chain hot chocolates with milk, cocoa powder, caramel syrup, vanilla extract, whipped cream, sprinkles, chocolate sauce, a flake, and a shot of espresso. Nonetheless, it all comes down to enumerating the first block, then considering the rows and columns stemming from that block, before completing it with the final four blocks and applying some group theory. Hopefully, it's all familiar stuff now.

There actually turns out to be 6 670 903 752 021 072 936 960 enumerations of the 9×9 Sudoku, according to a 2005 research project, and in 2006 it was found by mathematicians Bertram Felgenhauer and Frazer Jarvis in Sheffield that – by the conditions above – there are 5 472 730 538 *essentially different* enumerations of the 9×9 Sudoku grid.

If you're interested in reading more on either of the 4×4 and 9×9 Sudoku grids, including further study into other characteristics of the puzzle, some great places to start are Jason Rosenhouse and Laura Taalman's 2011 book *Taking Sudoku Seriously*, the write-up of Bertram Felgenhauer and Frazer Jarvis' findings from 2005, and Cornell University's 2009 *The Math Behind Sudoku* webpage. Wikipedia's *Mathematics of Sudoku* article gives great insight into some of the maths involved in other variations of the puzzle too.

3 Popular vs. Formal

> "*In our culture of mathematics, an all-too-common approach is to state the assumptions, state the theorems, prove the theorems, and stop. Any questions?*"
>
> - Strogatz (2014), *Writing about Math for the Perplexed and the Traumatized.*

The contrast between the popular-style article I have written in section 2 and a formal mathematical document – for example, Felgenhauer and Jarvis (2005) – is staggeringly obvious as the two contrasting styles seek to achieve different things. It is almost synonymous with how the weather forecast is presented: a popular article to a formal write-up is what the daily television weather-report is to the data analysed by the Met Office.

Strogatz (2014) recognises that formal mathematics seeks to "*state the assumptions, state the theorems, prove the theorems, and stop*", through depth, rigour and abstraction – so it is unnatural to dwell on topics for too long or write in prose throughout. It is usually a case of just stating key information and rigorous proofs, providing the reader a basis to develop a more complete understanding through their own work.

Conversely, a popular write-up aims to explain and expand upon complexities, going into much more detail with topics. Popular mathematics articles, for all intents and purposes, seek to be self-contained – without a huge amount of pre-requisite knowledge. For this reason, my article covers a range of mathematical topics which correlate to vastly different skill-levels. This is to ensure the reader has all of the relevant information.

For example, my article touches on the basics of factorials and – inexplicitly – the pigeonhole principle, which are widely considered very fundamental and basic mathematical ideas, but the article eventually leads to Group Theory, the Orbit-Stabiliser Theorem, and Burnside's Lemma – undergraduate level topics in mathematics. Thus, there is a huge contrast in the level of the mathematics used in the popular article – so much so that they may not typically appear together in an article of this length if it were written formally, but instead they may be referenced from another text. For example, Felgenhauer and Jarvis (2005) condenses the whole process of enumerating the first block into just half a page, with no introduction of the more basic concepts, such as factorials.

In my popular article, I also omitted in-line referencing throughout – something which would typically reduce the credibility of a formal paper. This is common within popular mathematics texts, for two main reasons. The first being that – as alluded to above – popular texts aim to be self-contained, and so readers should not be required to see where the information came from, as it should provide no relevant details that are not included in the popular text. The second reason is that referencing takes away from the natural flow of the article, and may thus deter the reader from reading on. In-line references are fundamentally irrelevant given the context of the document.

As a result, my article features a concluding paragraph with suggestions of further reading. This contains links to and details of the formal publications and webpages from which the information given in the article was taken – enabling correct attribution to the original sources and authors without disruption to the flow of the article.

The flow of the article is integral in allowing the reader to understand the concepts presented. For this reason, a number of features are present in my article which would not be typical in a formal paper. As is utilised to an extreme by Flannery (2006), my article features a running dialogue which takes the form of subheadings. These are present to guide the thoughts of the reader and to ensure that the questions they ask and the thoughts they are having directly relate to the body of text that follows and the information they will gain from that. This aims to keep the reader engaged, as it can often be exhausting to follow someone else's train of thought (Strogatz, 2014).

Another technique utilised to keep the reader engaged is presented by Greene (2013), involving the ordering and presenting of old and new information throughout the text.

In formal mathematics, statements are often made in isolation – with very little cross-referencing, as these links are often left to be made by the reader. In popular mathematics, however, information is presented in sentences and paragraphs, so is strongly connected. Moreover, it will be explicitly stated if new information relies on information previously given. I have done this in a way recommended by Greene (2013) which involves putting the old information at the beginning of sentences and paragraphs, and the new information towards the end – allowing the reader to recognise what subject is being talked about and to focus on what I am telling them *about* that subject.

To further enable a more natural flow and informal feel to the article, I have also taken into consideration the visual aesthetic and layout of the article and its elements. However, popular mathematics books are designed to be reader-friendly and thus much more attention is paid to the layout. In my article, by the recommendation of Strogatz (2014), I have arranged the layout of each page to be more dynamic. Most figures are cartoon-like and hand-drawn or feature some colour. Additionally, most are positioned to the side of the pages to allow the text they relate to to wrap around them – alleviating the need for the reader to search around for figures which may be referenced across the page using a number system, which is typical in a formal document. Similarly to the figures, equations are not labelled either – they are simply included in-line with the section of text which related to and describes them.

Finally, a key difference integrated into my popular mathematics article is a more beginner-friendly use of language, when compared to a formal document like that by Felgenhauer and Jarvis (2005). For example, my article quite heavily relies on analogies to better illustrate concepts – such as the symmetries in the dihedral group D_4 which was explained by the rotation of a biscuit. My article also utilises more natural language to refer to the blocks of the Sudoku grid, which are more easily understood by the reader to prevent having to refer back to previous pages. When referring to the blocks, I used language like “top-left”, or “bottom right”, whereas the labelling system of Felgenhauer and Jarvis (2005) would refer to these as $B1$ and $B4$.

Overall, there exists a vast range of differences between my popular-style article on the enumeration of Sudoku puzzles, and a formal document on the same topic – such as Felgenhauer and Jarvis (2005) – which favours abstraction and conciseness over the analogies and discursive explanations offered in my article (and most other typical popular mathematics articles). Both popular and formal mathematics exist to serve differing purposes and differing audiences, and it is for this reason that they are so contrasting. Strogatz (2014) and Greene (2013) provide some great further details of this contrast between formal and popular mathematics.

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